An Efficiency Case for Equity-Based School Priorities*

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Abstract

Many school districts operate "school choice" or "open enrollment" programs that give parents a choice of school. The popular schools in these districts are often oversubscribed, such that districts must decide which applicants receive priority at these schools. Typically, U.S. school districts give priority to students that live close to these schools or allocate by random lottery. However, another option used by a growing number of districts are priorities for disadvantaged students, which are intended to promote equity and reduce school segregation. This paper shows that, despite their effects on transportation costs, equity priorities can increase efficiency in the sense of raising aggregate welfare. They do this by facilitating better matches of students to schools. (JEL H73, H75, I21, I28)

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1 Introduction

Many school districts operate "school choice" or "open enrollment" programs that give parents a choice of school. In principle, choice-based student assignment offers three advantages over a traditional policy of neighborhood assignment.[1] First, it can improve overall educational quality by stimulating competition between schools (i.e., increase productive efficiency). Second, it can help parents to find schools that are a good match for their child (i.e., increase allocative efficiency). Third, it can advance equity and reduce school segregation by expanding access to good schools. In practice, school capacity is often constrained, such that not all students can be assigned to their preferred school. While capacity constraints limit the extent to which choice can improve productive efficiency (Hoxby, 2006; Neal, 2018), choice could still advance equity and improve allocative efficiency. Whether it does will depend crucially on how districts prioritize applicants to oversubscribed schools.

As we document below, districts prioritize applicants in various ways. Some use "neighborhood priorities" that favor students living close to the school. Others prioritize by random lottery. A third option used by a growing number of districts are priorities for disadvantaged students. Some of these are unconditional priorities (i.e., independent of the number of students who benefit from them), others are part of quota or reserve systems (often known as "controlled school choice"). Priorities for disadvantaged students are intended to promote equity and reduce school segregation and hence we refer to them as "equity priorities".

Since oversubscribed schools are often in affluent neighborhoods, equity priorities will likely increase transport costs and, one might assume, reduce overall welfare. This suggests that policy-makers that use equity priorities do so at the expense of welfare (i.e., an equity-efficiency trade-off). However, if parents' preferences are heterogeneous, such that the utility of attending a school equals the average utility across all parents plus some idiosyncratic match component, then an efficiency analysis must also account for whether good matches are realized. This paper points out that despite their effects on transport costs, equity priorities can increase welfare by facilitating better matches of students to schools. In other words, we present an efficiency case for equity priorities.

Our argument presumes a world in which i) parents incur additional transport costs from using a public school not in their neighborhood, ii) parents have idiosyncratic preferences over public schools in their area, iii) parents care about the socio-economic background of their child's peers, and iv) public schools have capacity constraints.[2] The starting point for the

[1] By "neighborhood assignment" we mean a system in which students are assigned to the school serving the attendance zone in which they live.

[2] There is ample evidence for transport costs, idiosyncratic preferences (Hastings et al., 2009; Abdulkadiroğlu

argument is that when parents have preferences over their child's peers, their school choices will be distorted. That is because they will be based partly on peer quality, which is zero-sum from the district perspective. Equity priorities can decrease peer differences across schools and thereby incentivize parents to put more weight on fundamentals (i.e., match quality) when choosing schools. This can increase aggregate welfare.

We use the simplest possible model to make the point. A community is divided into two equally-sized neighborhoods, each containing one school. Each school can accommodate one half of the student population. One neighborhood has higher socio-economic status households than the other. The schools differ in an exogenous way that makes one school a better match for some students than others. Parents know and care about these matching benefits but also have preferences over the socio-economic status of their child's classmates.[3] Households face a transport cost if their child attends the school not in their neighborhood. A school district assigns students to schools in the community. The district cannot observe students' matching benefits from the two schools and hence cannot simply assign students to their best match. Instead, it must respect incentive compatibility constraints that require that households find it in their interests to take the assignments intended for them.

We show that the optimal assignment can be implemented by one of three methods: neighborhood assignment (i.e., students are assigned to their neighborhood schools), school choice with neighborhood priorities, or school choice with equity priorities. With equity priorities, priority in assignments to the school in the more affluent neighborhood is given to households from the less affluent neighborhood. With neighborhood priorities, priority is given to households from the more affluent neighborhood. In both cases, available seats for non-priority households are randomly assigned to interested households.

Neighborhood assignment implements the optimal assignment when match benefits are small relative to transport costs and neighborhood inequality (i.e., the difference between the socio-economic status of the two neighborhoods). School choice is required when match benefits are large relative to transport costs and neighborhood inequality. Choice uses equity priorities when neighborhood inequality is high relative to transport costs. The intuition is that absent equity priorities, highly unequal neighborhoods yield highly segregated schools. As noted above, in the presence of peer preferences, segregated schools distort parents' school choices (i.e., inhibit their search for a good match). Equity priorities help to reduce school segregation and thereby facilitate better matches of students to schools.

Our focus on peer preferences connects our paper to other models of student assignment

et al., 2017) and peer preferences (Rothstein, 2006; Abdulkadiroğlu et al., 2020).

[3]We assume that these preferences do not vary by household type, such that preference heterogeneity is captured entirely by match benefits. We discuss this assumption in Section 4.

that assume peer effects (Arnott and Rowse, 1987; De Bartolome, 1990), as well as other models of school choice that assume peer preferences (Barseghyan et al., 2019; Allende, 2019; Avery and Pathak, 2021). It separates our paper from the large "school matching" literature that is concerned with school assignment but abstracts from peer preferences (see Pathak (2017) and Abdulkadiroglu and Andersson (2022) for surveys).[4] Our two-school model reinforces this separation, since it yields optimal assignments that can be implemented with simple school choice mechanisms, and hence we sidestep issues that would arise in models with three or more schools and that would require consideration of the types of mechanisms analyzed in the school matching literature (e.g., Deferred Acceptance, Boston Mechanism).

As noted above, some equity priorities fall under the "controlled school choice" umbrella (e.g., quota and reserve systems). Inspired by these types of systems, a strand of the school matching literature analyzes whether mechanisms that target diversity objectives can satisfy other desirable properties, notably "stability".[5] Our paper differs from this work in three key ways. First, since our model yields optimal assignments that can be implemented with simple school choice mechanisms that feature a simple equity priority (i.e., priority to residents of the disadvantaged neighborhood), we abstract from more complex priority structures. Second, while segregation turns out to be a central mechanism linking school priorities to welfare, we assume the social planner cares only about welfare. Third, as emphasized above, we allow for peer preferences. Indeed, segregation matters for welfare because, in the presence of peer preferences, it distort parents' school choices. To summarize, while our model features a simple equity priority, we suspect that our efficiency case for equity priorities applies to a broader class of priorities that includes controlled school choice.[6]

The rest of the paper is organized as follows. Section 2 provides empirical background. Section 3 presents our model, states its solutions, explains how it is solved, and discusses the underlying economic intuition. Section 4 concludes.

[4]As Abdulkadiroglu and Andersson (2022) note, "In general, families care about schools as well as peers of their children at schools. However, it is difficult, and most of the time impossible, to extend the theory of matching by generalizing student preferences over sets of students enrolled at schools." (p.6) See Cox et al. (2021) and Leshno (2021) for recent attempts to incorporate peer preferences into matching mechanisms.

[5]Stability (also known as "absence of justified envy") is violated when a student is assigned to a school and another student is not assigned despite having priority over the assigned student and despite preferring the school to their assigned school. Abdulkadiroglu and Andersson (2022) review this literature (section 8.1).

[6]Shi (2022) considers optimal (i.e., welfare-maximizing) priority design in a setup with flexible priorities (drawn from a probability distribution) but without peer preferences.

2 Choice, capacity constraints and school priorities

To motivate our analysis, this section documents the prevalence of choice, the existence of capacity constraints and various school priorities used by choice districts in the U.S.

2.1 School choice

A survey of American parents conducted in 2016 revealed that in more than 30% of cases, their children did not attend their "regularly assigned" school.[7] This suggests that many parents exercise some form of school choice, but does not reveal the precise form that this choice takes. Using data from Whitehurst (2017), which documents the student assignment methods used by the 112 largest school districts in the U.S., we classify these methods as either "neighborhood assignment", "opt-out choice" or "centralized choice".[8] Under "neighborhood assignment" (49 districts and 37% of enrollment) students attend their attendance-zone school. Under "opt-out choice" (51 districts and 47% of enrollment), students are defaulted to their attendance-zone school but can express a preference for - and attend - another school. Under "centralized choice" (12 districts and 16% enrollment) there is no default school. Instead, parents rank schools and students are assigned via an algorithm that accounts for their rankings and their priorities at different schools.

2.2 Capacity constraints

Districts that use choice-based systems do not guarantee that students will be assigned to their preferred school. Capacity constraints are difficult to quantify, but centralized choice systems yield one measure: the fraction of parents not assigned to their first-choice school. For one half of the centralized choice districts listed in Table 1, this exceeds 50%.

2.3 School priorities

In the presence of capacity constraints, choice districts must decide which students receive priority at oversubscribed schools. By definition, opt-out choice districts use neighborhood

[7] See Figure 1 of Wang et al. (2019).

[8] Our classification is based on criteria used by Whitehurst (2017). Centralized-choice districts are those coded "Yes" to Question 5A: *"Students assigned to schools through an application process in which parents express their preferences (rather than through geographical attendance zones)"*. Opt-out choice districts are those coded "Yes" to Question 5B: *"Students receive default school assignment based on geographical attendance zone but parents can easily express their preferences for other schools"*. Neighborhood-assignment districts are those coded "Yes" to Question 5G: *"Assignment to schools out of student's geographical attendance zone is impossible or difficult"*.

priorities (i.e., prioritize attendance-zone students). In contrast, different centralized choice districts use different methods. To provide a flavor of this variation, note that the 10 districts listed in Table 1 use one of three main methods.[9] First, some districts use various types of neighborhood priorities. To the extent that popular schools are found in more affluent neighborhoods, these priorities imply that more-advantaged families will have priority at popular schools. Second, some districts use random lotteries.[10] Third, some districts use equity priorities that favor less-advantaged students. Different districts do this in different ways, but the underlying goal is the same: to enable less-advantaged students to attend oversubscribed schools.[11]

Table 1 Capacity Constraints and School Priorities

	Not assigned to 1st-choice	Main method of rationing
Baltimore (DC)		
Middle Schools	56%	Neighborhood
High Schools	50%	Lottery
Boston (MA)		
Pre-2013	55%	Neighborhood
Post-2013	Not available	Lottery
Denver (CO)	19%	Reduced-Price Lunch
Lee (FL)	15%	Neighborhood
Milwaukee (WI)	5%	Neighborhood
New York City (NY)	68%	Varies (see text)
Newark City (NJ)	47%	Neighborhood
Oakland Unified (CA)	34%	Neighborhood
Orleans Parish (LA)	55%	Neighborhood
San Francisco (CA)	37%	Census Tract of Residence

Notes: The table contains information for 10 of the 12 districts classified as "centralized choice" using the Whitehurst (2017) data (see text for details). The second column reports the fraction of applicants assigned to a school other than the first-choice school (in year for which data most recently available). The third column describes the main method of rationing (i.e., the students prioritized for seats that remain after continuing students and siblings assigned.)

[9] These districts give first priority to continuing students and students with siblings already enrolled at a school, hence "main method" is the first method used to allocate remaining seats.

[10] Nearly all districts use lotteries after other priorities have been applied.

[11] In Denver and San Francisco, disadvantaged students receive priority over all seats (where applicable). In New York City, a proportion of seats are reserved for disadvantaged students defined by family income and other criteria (Margolis et al., 2023; Idoux, 2022).

3 Model and results

3.1 The model

The model features a single community with a population of households of size 2. The community is divided into two neighborhoods, A and B, each containing $1/2$ of the population. There are two schools serving the community, one in each neighborhood. Each school has a capacity of 1. The school in neighborhood $J \in \{A, B\}$ is referred to as school J.

Households differ in their socio-economic status. The neighborhoods are stratified and A is the more affluent neighborhood. Neighborhood A consists of households of socio-economic status μ, while neighborhood B is comprised of households of status $-\mu$. The parameter μ measures the degree of neighborhood inequality.

Each household has a child that must attend one of the two schools. Households care about peer quality at the school their child attends. This is measured by the average socio-economic status of the school's students.

Households also have idiosyncratic preferences for the two schools that reflect the quality of the perceived match between the school and their child. To keep household preference types one dimensional, we assume that households only differ in their preferences for school A. Specifically, all households obtain a match benefit 0 from school B and an idiosyncratic match benefit m from school A. The school A match benefit m is uniformly distributed on $[-M, M]$. The parameter M measures the extent of potential match benefits. Households incur a "transport cost" c if using the school not in their neighborhood. This cost captures the additional transaction costs arising from using this school.

Let s_J denote the average socio-economic status of school J's students. Then, a household living in neighborhood A with match benefit m obtains a payoff $s_A + m$ from using school A and a payoff $s_B - c$ from school B. A household living in neighborhood B with match benefit m obtains a payoff s_B from using school B and a payoff $s_A + m - c$ from school A.

3.2 The school assignment problem

A school district is tasked with assigning the community's children to the two schools. Each household is characterized by its location J and its match benefit m. An *assignment* is a pair of functions $(\pi_A(m), \pi_B(m))$ mapping points in $[-M, M]$ to points in the interval $[0, 1]$. The interpretation is that the child of a household in neighborhood J with match benefit m is assigned to school A with probability $\pi_J(m)$ and school B with probability $1 - \pi_J(m)$. An assignment is *feasible* if $1/2$ the community's students are assigned to each school. This requires that

$$\int_{-M}^{M} (\pi_A(m) + \pi_B(m)) \frac{dm}{2M} = 1. \tag{1}$$

Under the assignment $(\pi_A(m), \pi_B(m))$, the average socio-economic status of the two schools are

$$s_A = \mu \int_{-M}^{M} (\pi_A(m) - \pi_B(m)) \frac{dm}{2M}, \tag{2}$$

and

$$s_B = \mu \int_{-M}^{M} (\pi_B(m) - \pi_A(m)) \frac{dm}{2M}. \tag{3}$$

The expected payoff of a neighborhood A household with match benefit m is

$$\pi_A(m) (s_A + m) + (1 - \pi_A(m)) (s_B - c) \tag{4}$$

and the expected payoff of a neighborhood B household with match benefit m is

$$\pi_B(m) (s_A + m - c) + (1 - \pi_B(m)) s_B. \tag{5}$$

The school district's objective function is the sum of households' expected payoffs. This objective function can be simplified by observing that the peer quality effects are zero sum and thus wash out. This can be seen from (2) and (3), which imply that s_A and s_B must sum to zero. This implies that, if the assignment $(\pi_A(m), \pi_B(m))$ is feasible, the school district's objective function is equal to

$$W = \int_{-M}^{M} [m\pi_A(m) + m\pi_B(m) - c(1 - \pi_A(m)) - c\pi_B(m)] \frac{dm}{2M}. \tag{6}$$

Thus, the school district's payoff depends only on the match benefits and the transport costs.

3.3 The first best

If the school district could observe each household's match quality m, its problem would be to choose an assignment $(\pi_A(m), \pi_B(m))$ to maximize the objective function (6) subject to the feasibility constraint (1). The *first best assignment* is the assignment that solves this problem.

The first best assignment has a very simple form: neighborhood A households are assigned to school A if their match benefits exceed $-c$, while neighborhood B households are assigned to school A if their match benefits exceed c. This is established in:

Proposition 1 *The first best assignment is*

$$\pi_A(m) = \begin{cases} 1 \text{ } if \text{ } m > -c \\ 0 \text{ } if \text{ } m < -c \end{cases}, \tag{7}$$

and

$$\pi_B(m) = \begin{cases} 1 \text{ } if \text{ } m > c \\ 0 \text{ } if \text{ } m < c \end{cases}. \tag{8}$$

3.4 The second best

When the school district cannot observe each household's match benefit m, it must respect incentive compatibility constraints that ensure that households take the assignments intended for their children. For households in neighborhood A, these constraints are as follows: for all $m \in [-M, M]$ and any $m' \in [-M, M]$

$$\pi_A(m)(s_A + m) + (1 - \pi_A(m))(s_B - c) \geq \pi_A(m')(s_A + m) + (1 - \pi_A(m'))(s_B - c). \tag{9}$$

Similarly, for households in neighborhood B, the constraints are: for all $m \in [-M, M]$ and any $m' \in [-M, M]$

$$\pi_B(m)(s_A + m - c) + (1 - \pi_B(m))s_B \geq \pi_B(m')(s_A + m - c) + (1 - \pi_B(m'))s_B. \tag{10}$$

Note that the peer qualities s_A and s_B appear in the incentive compatibility constraints and depend on the assignment via the equations (2) and (3).

An assignment $(\pi_A(m), \pi_B(m))$ is *incentive compatible* if it satisfies the incentive compatibility constraints (9) and (10) when s_A and s_B satisfy the equations (2) and (3). The second best problem is then to choose an incentive compatible and feasible assignment that maximizes the school district's objective function (6). A *second best assignment* is an assignment that solves this problem.

3.4.1 The second best assignment when $M \leq c$

When considering the second best assignment, the first question to ask is whether the first best assignment is incentive compatible. If so, this is also the second best assignment. When M is less than c, the match benefits are small relative to the transport costs and the first best assignment just involves assigning students to their neighborhood schools. We refer to this as *neighborhood assignment*. This assignment is clearly incentive compatible because all students in each neighborhood are treated in a uniform way. Thus we have:

Proposition 2 *If $M \leq c$, the second best assignment is neighborhood assignment; i.e.,* $(\pi_A(m), \pi_B(m)) = (1, 0)$ *for all m.*

3.4.2 The second best assignment when $M > c$

When M exceeds c, the match benefits are large relative to the transport costs and the first best assignment is not incentive compatible if there is neighborhood inequality (i.e., if $\mu > 0$). To see this, note that under the first best assignment, the peer quality in school A, s_A, is $\mu \Pr(m \geq -c) - \mu \Pr(m \geq c)$ and the peer quality in school B, s_B, is $\mu \Pr(m < -c) - \mu \Pr(m < c)$. Now consider a household living in neighborhood A of type $-(c+\varepsilon)$ where ε is small. If they report their type accurately, their child is assigned to school B and they get a payoff $s_B - c$. If they report their type to be $-c$, they get a payoff $s_A - (c+\varepsilon)$. They gain from misreporting if the *peer quality difference*, defined to be $\Delta s \equiv s_A - s_B$, exceeds ε. Subtracting s_B from s_A, we see that Δs is equal to $2\mu c/M$. Thus, for ε sufficiently small, misreporting is optimal. It follows that the second best assignment will not equal the first best assignment when M exceeds c.

What does the second best assignment look like in this case? The answer is not obvious. One question is whether neighborhood assignment could be optimal. While this minimizes transport costs, when M exceeds c, it seems potentially desirable to assign some households to non-neighborhood schools. However, we would expect school A to be more attractive than school B (i.e., we would expect the peer quality difference to be positive), such that not every household that prefers school A can be assigned to school A. This begs a second question of which students should be assigned. For example, would it be better to assign a random subset of those that prefer school A or favor students from the poorer neighborhood? Favoring students from the poorer neighborhood would increase transport costs, but should help to reduce the peer quality difference between the schools and thereby reduce school A's advantage. A third question is whether it could even be optimal to assign students from the poorer neighborhood to the school in the richer neighborhood when their parents prefer the local school? Such a strategy might be justified on the grounds of reducing the peer quality difference.

Our next proposition describes the second best assignment when M exceeds c. The solution is relatively simple. The peer quality difference does end up being positive, so school A has an advantage. Only households who prefer school A are assigned to it, but not all who prefer it are so assigned. In rationing access, priority is either given to interested neighborhood A households or interested neighborhood B households. Which of these possibilities arises depends on the three parameters μ, M, and c.

Proposition 3 i) *Suppose that $M > c$ and that $\mu < c/2$. Then, if $M \leq c + 2\mu$, the second best assignment is neighborhood assignment (i.e., $(\pi_A(m), \pi_B(m)) = (1, 0)$ for all m). If $M > c + 2\mu$, the second best assignment is*

$$\pi_A(m) = \begin{cases} 1 \text{ if } m \geq -c - \frac{2\mu c}{M-2\mu} \\ 0 \text{ if } m < -c - \frac{2\mu c}{M-2\mu} \end{cases}, \tag{11}$$

and

$$\pi_B(m) = \begin{cases} \frac{M-c-\frac{2\mu c}{M-2\mu}}{M-c+\frac{2\mu c}{M-2\mu}} \text{ if } m \geq c - \frac{2\mu c}{M-2\mu} \\ 0 \text{ if } m < c - \frac{2\mu c}{M-2\mu} \end{cases}. \tag{12}$$

This assignment can be implemented by school choice, with priority at school A given to neighborhood A households.

ii) *Suppose that $M > c$ and that $\mu > c/2$. Then, if $M < c + \sqrt{\mu^2 + 4c\mu} - \mu$, the second best assignment is neighborhood assignment. If $M > c + \sqrt{\mu^2 + 4c\mu} - \mu$, the second best assignment is*

$$\pi_A(m) = \begin{cases} \frac{M+c-\frac{2\mu c}{M+2\mu}}{M+c+\frac{2\mu c}{M+2\mu}} \text{ if } m \geq -c - \frac{2\mu c}{M+2\mu} \\ 0 \text{ if } m < -c - \frac{2\mu c}{M+2\mu} \end{cases}, \tag{13}$$

and

$$\pi_B(m) = \begin{cases} 1 \text{ if } m \geq c - \frac{2\mu c}{M+2\mu} \\ 0 \text{ if } m < c - \frac{2\mu c}{M+2\mu} \end{cases}. \tag{14}$$

This assignment can be implemented by school choice, with priority at school A given to neighborhood B households.

Part i) of the Proposition distinguishes two sub-cases when μ is less than $c/2$. If M is less than $c + 2\mu$, students are assigned to their neighborhood school. This assignment yields a peer quality difference equal to 2μ. Given this, and the fact that $2\mu - M$ exceeds $-c$, all neighborhood A households prefer school A. Some neighborhood B households also prefer school A, but they are not so assigned.

If M exceeds $2\mu + c$, the assignment is described by equations (11) and (12). The peer quality difference Δs in this assignment is $2\mu c/(M - 2\mu)$. The assignment is incentive compatible because the only neighborhood A households assigned to school A are those for whom $m + \Delta s$ exceeds $-c$ and the only neighborhood B households assigned to school A are those for whom $m + \Delta s$ exceeds c. However, there is an important difference between neighborhood A and B households: *all* neighborhood A households who prefer school A are assigned to it, while only *some* of neighborhood B households who prefer school A are assigned to it.

The optimal assignment in this second sub-case can be implemented by a school choice program. The school district simply asks all households to choose their preferred school. All neighborhood A households choosing school A and all households choosing school B are assigned to their preferred schools. The seats at school A not wanted by neighborhood A households are assigned randomly among the neighborhood B households who have chosen school A; some will have to attend school B. In this program, therefore, priority at the oversubscribed school is given to those who live in that school's neighborhood (neighborhood priorities).

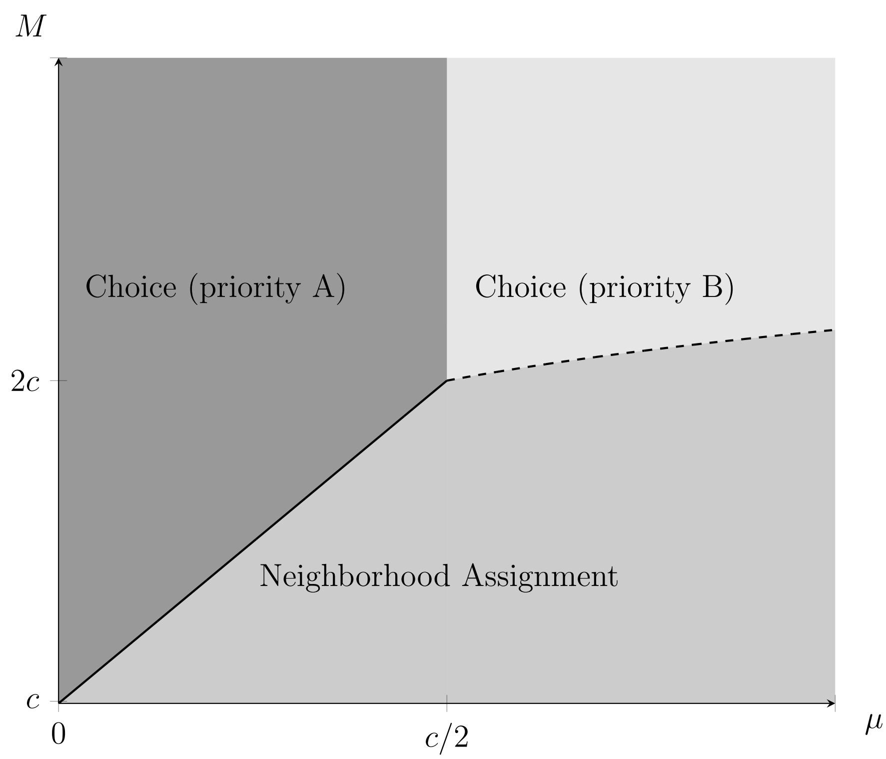
Part ii) of the Proposition distinguishes two sub-cases when μ exceeds $c/2$. If M is less than $c + \sqrt{\mu^2 + 4c\mu} - \mu$, students are assigned to their neighborhood school. As explained above, this assignment leads to a peer quality difference of 2μ. Note that when μ exceeds $c/2$, $c + \sqrt{\mu^2 + 4c\mu} - \mu$ must be less than $c + 2\mu$. Accordingly, $2\mu - M$ exceeds $-c$ and all neighborhood A households prefer school A. Some neighborhood B households prefer school A, but again they are not so assigned.

If M exceeds $c + \sqrt{\mu^2 + 4c\mu} - \mu$, the assignment is described by equations (13) and (14). The peer quality difference Δs in this assignment is equal to $2\mu c/\left(M + 2\mu\right)$. The assignment is incentive compatible because the only neighborhood A households assigned to school A are those for whom $m + \Delta s$ exceeds $-c$ and the only neighborhood B households assigned to school A are those for whom $m + \Delta s$ exceeds c. However, the priorities are reversed from the analogous case in part i): now *all* neighborhood B households who prefer school A are assigned to it, while only *some* of neighborhood B households who prefer school A are assigned to it.

Again, the optimal assignment in the second sub-case can be implemented by a school choice program. The school district asks all households to choose their preferred school. All neighborhood B households choosing school A and all households choosing school B are assigned to their preferred school. The seats at school A not wanted by neighborhood B households are assigned randomly among the neighborhood A households who have chosen school A. In this program, therefore, priority at the oversubscribed school is given to those who live in the less-affluent neighborhood (equity priorities).

Figure 1 illustrates the solution for fixed c as a function of the parameters μ and M. The left hand side of the figure illustrates part i) of the Proposition (when μ is less than $c/2$). The solid black line describes the function $M = c + 2\mu$. When M is below this line, the district should use neighborhood assignment. When M is above the line, the district should allow school choice, but give priority to neighborhood A students in assigning to school A. The right hand side of the figure illustrates part ii) of the Proposition (when μ exceeds $c/2$). The dashed black line describes the function $M = c + \sqrt{\mu^2 + 4c\mu} - \mu$. When M is below this line, the district should again use neighborhood assignment. When M is above the line, the district should allow school choice, but give priority to neighborhood B students in assigning to school A.

Figure 1 Illustration of Proposition 3



3.4.3 Proving Proposition 3

While our model is spartan and the second-best assignment is simple to understand, establishing the properties of this assignment is challenging. This sub-section explains the key steps of the proof; full details can be found in the Online Appendix.

Step 1. The first step is to simplify the incentive compatibility constraints. Starting with neighborhood A households, if $s_A + m$ exceeds $s_B - c$, so that the household prefers school A, then the household will want to report an m' that maximizes $\pi_A(m')$ - the probability they are assigned to school A. Thus, incentive compatibility requires that $\pi_A(m)$ equal

$\max \pi_A(m')$. If $s_A + m$ is less than $s_B - c$, so that the household prefers school B, then the household will want to report an m' that minimizes $\pi_A(m')$. Thus, incentive compatibility requires that $\pi_A(m)$ equal $\min \pi_A(m')$. By a similar logic, if $s_A + m$ is less than $s_B - c$, so that a neighborhood B household prefers school B, incentive compatibility requires that $\pi_B(m)$ equal $\min \pi_B(m')$. If $s_A + m - c$ exceeds s_B, so that a neighborhood B household prefers school B, incentive compatibility requires that $\pi_B(m)$ equal $\min \pi_B(m')$.

It follows that incentive compatible assignments can be characterized by four probabilities $(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$, which must satisfy $0 \leq \underline{\pi}_J \leq \overline{\pi}_J \leq 1$, along with an associated peer quality difference Δs. The probability $\overline{\pi}_J$ is the probability that J households who prefer A are assigned to school A and the probability $\underline{\pi}_J$ is the probability that J households who prefer school B are assigned to school A. The associated peer quality difference Δs is that generated by these assignments.

Step 2. The second step is to reduce the set of incentive compatible assignments that need to be considered. Our first observation is that, in a second best assignment, $\underline{\pi}_J$ must equal 0 if $\overline{\pi}_J$ is less than 1. Otherwise, some neighborhood J households with students assigned to school A would prefer school B, while some with students assigned to school B would prefer school A. It would be better to substitute those who prefer school A for those who do not.

Our second observation is that, in a second best assignment, the peer quality difference Δs cannot be negative. Intuitively, a negative peer quality difference would create a cross-hauling inefficiency, since it would require more than 50% of A households to attend school B, with an equal fraction of B households traveling the other way to school A. To prove that the peer quality difference cannot be negative, we show that any incentive compatible and feasible assignment with negative Δs can be dominated by either neighborhood assignment or what we call the *zero peer quality difference assignment*. Under the latter assignment, students from A households that prefer school B given a zero peer quality difference are assigned to school B with probability 1; the other A students are assigned to school A with probability $M/(m + c)$; students from B households that prefer school A are assigned to school A with probability 1; the other B students are assigned to school A with probability $c/(M+c)$. This assignment achieves an equal mix of A and B households in both schools and thereby eliminates the peer quality difference between them. Crucially, it does this by forcing some students to attend the non-neighborhood school despite those households preferring the neighborhood school.

The observation that Δs must be non-negative implies that in any second best assignment with non-zero Δs, school A will be in excess demand. That follows because $M + \Delta s$ must exceed c, such that some fraction of neighborhood B households prefer school A, and this fraction will exceed the fraction of neighborhood A households that prefer school B.

Step 3. The third step is to shed further light on the $\underline{\pi}_J$'s. In particular, having shown that $\underline{\pi}_J$ equals 0 if $\overline{\pi}_J$ is less than 1, we show that $\underline{\pi}_J$ equals 0 even if $\overline{\pi}_J$ equals 1. The result is obvious for neighborhood A households. After all, the underlying incentive problem is that because of the peer difference, too few of these households prefer school B. It cannot then make sense to assign A households that prefer school B to school A. It is better to replace these students with neighborhood B students that prefer school A.

The result is less obvious for neighborhood B households. Intuitively, to reduce the peer difference, one might assign B students to school A even if those households prefer school B. This is the logic behind the zero peer quality difference assignment and, under some conditions, the school district will prefer this to neighborhood assignment.[12] It turns out, however, that this is never optimal. To establish this we show that starting with any assignment in which $\overline{\pi}_B$ equals 1 and $\underline{\pi}_B$ is positive, we can increase the school district's payoff by reducing $\underline{\pi}_B$ to zero and raising $\overline{\pi}_A$ to keep school A full.[13] We demonstrate this by showing that the gains to households who benefit from the change exceed the losses to households who lose.

Step 4. The fourth step is to turn attention to the $\overline{\pi}_J$'s. Our main result is that the second best assignment *either* admits all A children whose parents prefer school A *or* admits all B children whose parents prefer school A. To prove this result we show that any assignment with neither $\overline{\pi}_A$ nor $\overline{\pi}_B$ equal to 1 can be changed to increase the school district's payoff. The nature of the change depends on whether the initial assignment features A households that prefer school B. If not, the initial assignment can be dominated by neighborhood assignment. If so, it can be dominated by an assignment in which $\overline{\pi}_B$ is marginally decreased if μ is less than $c/2$ and increased if μ exceeds $c/2$. The key implication of this step is that it rules out the possibility that seats in school A are randomly allocated among interested households.

Step 5. Having established that the second best assignment involves either admitting all A children whose parents prefer school A or admitting all B children whose parents prefer school A, the fifth step is to understand other features of these assignments. In particular, for a second best assignment in which $\overline{\pi}_A$ equals 1, we describe what $\overline{\pi}_B$ and Δs must look like. We refer to the resulting assignment as an A *priority assignment*. Similarly, we describe what $\overline{\pi}_A$ and Δs must look like for second best assignments in which $\overline{\pi}_B$ equals 1 and refer to the resulting assignment as a B *priority assignment*. We also describe the payoffs the school

[12] It generates a higher payoff when M exceeds $3c$.

[13] More precisely, starting with any assignment that has those properties and is also feasible, incentive compatible and has non-negative Δs.

district obtains from A and B priority assignments.

Step 6. The sixth and final step is to compare the payoffs generated by A and B priority assignments and identify the conditions under which one dominates the other. Understanding this, along with our knowledge of what these assignments look like, yields Proposition 3.

3.4.4 Optimal priorities under choice

Having stated Proposition 3 and outlined the proof, we now provide intuition for our key result about optimal priorities under choice: that B households should receive priority at school A when neighborhood inequality is large relative to transport costs. We consider parameter values for which choice is optimal and study whether priority at school A should be given to neighborhood A or B households.

To begin, note that we can write welfare under an A priority assignment as $W_A(\theta_A) = \theta_A(M+c) - 2\theta_A c$, where $\theta_A = \frac{M-(c+\triangle s_A)}{2M}$ is the fraction of neighborhood A students that attend school B and $\triangle s_A = \frac{2\mu c}{M-2\mu}$ is the equilibrium peer difference given A priority.[14] Similarly, we can write welfare under a B priority assignment as $W_B(\theta_B) = (M-c)(1-\theta_B) - 2\theta_B c$, where $\theta_B = \frac{M-(c-\triangle s_B)}{2M}$ and $\triangle s_B = \frac{M-(c-\triangle s_A)}{2M}$.[15] Given parameter values (M, c) the "switching rates" θ_A and θ_B are sufficient statistics for the two components of welfare associated with an assignment: the average match value (the first term in these welfare expressions) and the average transport cost (the second term). The "optimal switching rates" (i.e., that yield the first-best assignment) are $\theta_A = \theta_B \equiv \theta^* = \frac{M-c}{2M}$, the fraction of students better matched to the non-neighborhood school. It follows that $W_A(\theta^*) = W_B(\theta^*) \equiv W^* = \frac{(M-c)^2}{2M}$, where W^* is welfare under the first-best assignment.

When μ equals 0, both $\triangle s_A$ and $\triangle s_B$ equal zero and θ_A and θ_B equal θ^*. However, when μ exceeds 0, then both $\triangle s_A$ and $\triangle s_B$ are positive and we have that θ_A is less than θ^* and θ_B exceeds θ^*. In other words, when the neighborhoods are unequal, an A priority assignment is associated with "under-switching" (because neighborhood A students do not want to lose the peer difference) and a B priority assignment is associated with "over-switching" (because neighborhood B students want to gain the peer difference). We can write $W_A(\theta_A) = W^* - \widetilde{\theta}_A(M-c)$, where $\widetilde{\theta}_A \equiv \theta^* - \theta_A = \frac{\triangle s_A}{2M}$ is the extent of under-switching. Similarly, we can write $W_B(\theta_B) = W^* - \widetilde{\theta}_B(M+c)$, where $\widetilde{\theta}_B \equiv \theta_B - \theta^* = \frac{\triangle s_B}{2M}$ is the extent of over-switching. In both assignments, welfare decreases as we move away from the optimal switching rate (i.e., $W'_A(\tilde{\theta}_A), W'_B(\tilde{\theta}_B) < 0$). However, welfare decreases faster as we move away from the optimal switching rate in a B assignment. That is because over-switching increases transport costs,

[14]This follows from $W_A = \frac{(M-c)^2-(M-c)\triangle s_A}{2M}$ and $\triangle s_A = \frac{2\mu c}{M-2\mu}$ in equation (19) of the Online Appendix.
[15]This follows from $W_B = \frac{(M-c)^2-(M+c)\triangle s_B}{2M}$ and $\triangle s_B = \frac{2\mu c}{M+2\mu}$ in equation (22) of the Online Appendix.

while under-switching decreases transport costs.[16] This implies that if switching distortions are similar across the two priority regimes, then A priority is preferred on transport cost grounds. However, if switching rate distortions yield less over-switching in B priority than under-switching in A priority, then B priority could be preferred on the grounds that it yields better matches (despite higher transport costs). More formally, we see that B priority assignment is preferred when $\frac{\widetilde{\theta}_B}{\widetilde{\theta}_A} < \frac{M-c}{M+c}$, where the left-hand side is the ratio of over- to under-switching and the right-hand is a threshold that is below one and that depends on the scope for matching.

Since $\frac{\widetilde{\theta}_B}{\widetilde{\theta}_A} = \frac{\triangle s_B}{\triangle s_A} = \frac{M-2\mu}{M+2\mu}$, we obtain the result that B priority is preferred when μ exceeds $\frac{c}{2}$. But why is the ratio of over- to under-switching decreasing in μ? The key insight is that a B priority assignment imposes a natural brake on the process by which larger μ increases the extent of non-optimal switching. Specifically, as μ increases and a higher fraction of households switch from neighborhood B into school A, the peer difference increases by less than it would otherwise (because conditional on μ the composition of the two schools becomes more similar). There is no such brake in an A priority assignment. Instead, the reduction in switching out of neighborhood A that follows an increase in μ *magnifies* the resulting increase in the peer difference between the schools and hence further limits switching out of neighborhood A. This explains why higher levels of neighborhood inequality are associated with a lower ratio of over- to under-switching and hence a preference for B priority.

4 Conclusion

We have analyzed a central problem facing districts that operate school choice: how to prioritize applicants to oversubscribed schools. This problem is often framed in terms of an equity-efficiency trade-off: equity priorities can broaden access to oversubscribed schools, and reduce school segregation, but only by increasing transport costs and thereby reducing welfare. By assuming heterogeneous preferences, we added another dimension to this problem: the extent to which priorities facilitate good matches of students to schools. By assuming peer preferences, we ensured that this matching process was complicated by parents' desire to attend schools that enroll affluent students. Our model delivers what we believe is a novel insight: equity priorities can improve efficiency by limiting the extent of school segregation and thereby aiding parents' search for a good match.

While we expect that this insight will remain true in richer models, there is much scope

[16] Under- and over-switching reduces match values relative to the first-best assignment, and the reduction is smaller for over-switching, but this does not compensate for the increased transport costs.

for further analysis of the problem we have studied. For example, while our model assumed that transport costs were the same for both types of households, it would be interesting to relax that assumption, perhaps by allowing the school district to choose school priorities and transport subsidies. This is especially relevant given wide variation across U.S. school districts in the extent to which transport is subsidized (McShane and Shaw, 2020). One could also relax the assumption that peer preferences were the same for both types of households, although the implications here seem more obvious and less interesting. For example, if more-advantaged families had stronger peer preferences, then, ceteris paribus, welfare would be increasing in the extent of segregation and the case for equity priorities would be weaker.

It would also be interesting to extend the model to multiple schools and neighborhoods. It seems possible that the district would continue to choose between neighborhood priorities and an equity priority, but where the latter involved parents choosing schools in order of disadvantage (i.e., parents in the most-disadvantaged neighborhoods choose first). However, tackling the multiple school case would be considerably more challenging.

The model could also be extended to allow parents choose where to live.[17] One possibility is that equity priorities incentivize higher-income households to live in disadvantaged neighborhoods, thereby reducing residential segregation and further aiding the matching process. In contrast, it seems clear that extending the model to allow exit (e.g., to private schools) will weaken the case for equity priorities, since some more-advantaged families will prefer to exit than to attend unpopular schools, potentially increasing school segregation (Bibler and Billings, 2020; Idoux, 2022; Bjerre-Nielsen and Gandil, 2023). It would be interesting to analyze whether districts could combat exit by deploying auxiliary policies such as subsidized transport out of higher-income neighborhoods, special (e.g., magnet) programs in disadvantaged schools or property tax rebates for families with children in public schools.

[17] Some recent papers explore the interaction between school and residential choice (Agostinelli et al., 2023; Greaves and Turon, 2021; Grigoryan, 2021; Park and Hahm, 2023; Xu, 2019). This builds on some earlier work that considers more limited forms of choice but also allows for peer preferences (Epple and Romano, 2003).

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ONLINE APPENDICES

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Appendix A. Proof of Proposition 3

In the main paper we discussed how Proposition 3 was established, and highlighted the key steps of the proof. In this Appendix we provide a more formal exposition of this proof. This Appendix states several Lemmata. Appendix B contains the proofs of these results along with the proof of Proposition 1. Appendix C collects together the Figures referred to in Appendix B.

We begin by restating the equations in the main text (see the main text for more discussion and definitions). These equations are:

- Feasible assignment condition:

$$\int_{-M}^{M} (\pi_A(m) + \pi_B(m)) \frac{dm}{2M} = 1. \tag{1}$$

- Average socio-economic status of the two schools given assignment $(\pi_A(m), \pi_B(m))$:

$$s_A = \mu \int_{-M}^{M} (\pi_A(m) - \pi_B(m)) \frac{dm}{2M}, \tag{2}$$

  and

$$s_B = \mu \int_{-M}^{M} (\pi_B(m) - \pi_A(m)) \frac{dm}{2M}. \tag{3}$$

- Expected payoff to neighborhood A household with match benefit m:

$$\pi_A(m) (s_A + m) + (1 - \pi_A(m)) (s_B - c) \tag{4}$$

- Expected payoff to neighborhood B household with match benefit m:

$$\pi_A(m) (s_A + m - c) + (1 - \pi_A(m)) s_B. \tag{5}$$

- School district objective function:

$$W = \int_{-M}^{M} [m\pi_A(m) + m\pi_B(m) - c (1 - \pi_A(m)) - c\pi_B(m)] \frac{dm}{2M}. \tag{6}$$

- First best assignment:

$$\pi_A(m) = \begin{cases} 1 \ \textit{if} \ m > -c \\ 0 \ \textit{if} \ m < -c \end{cases}, \tag{7}$$

  and

$$\pi_B(m) = \begin{cases} 1 \ \textit{if} \ m > c \\ 0 \ \textit{if} \ m < c \end{cases}. \tag{8}$$

- Incentive compatibility constraint for households in neighborhood A, for all $m \in [-M, M]$ and any $m' \in [-M, M]$

$$\pi_A(m)\left(s_A + m\right) + \left(1 - \pi_A(m)\right)\left(s_B - c\right) \geq \pi_A(m')\left(s_A + m\right) + \left(1 - \pi_A(m')\right)\left(s_B - c\right). \tag{9}$$

- Incentive compatibility constraint for households in neighborhood B, for all $m \in [-M, M]$ and any $m' \in [-M, M]$

$$\pi_B(m)\left(s_A + m - c\right) + \left(1 - \pi_B(m)\right)s_B \geq \pi_B(m')\left(s_A + m - c\right) + \left(1 - \pi_B(m')\right)s_B. \tag{10}$$

- Second best assignment when $M > c$, $\mu > c/2$ and $M > c + 2\mu$

$$\pi_A(m) = \begin{cases} 1 \ \mathit{if} \ m \geq -c - \frac{2\mu c}{M-2\mu} \\ 0 \ \mathit{if} \ m < -c - \frac{2\mu c}{M-2\mu} \end{cases}, \tag{11}$$

and

$$\pi_B(m) = \begin{cases} \frac{M-c-\frac{2\mu c}{M-2\mu}}{M-c+\frac{2\mu c}{M-2\mu}} \ \mathit{if} \ m \geq c - \frac{2\mu c}{M-2\mu} \\ 0 \ \mathit{if} \ m < c - \frac{2\mu c}{M-2\mu} \end{cases}. \tag{12}$$

- Second best assignment when $M > c$, $\mu < c/2$ and $M > c + \sqrt{\mu^2 + 4c\mu} - \mu$

$$\pi_A(m) = \begin{cases} \frac{M+c-\frac{2\mu c}{M+2\mu}}{M+c+\frac{2\mu c}{M+2\mu}} \ \mathit{if} \ m \geq -c - \frac{2\mu c}{M+2\mu} \\ 0 \ \mathit{if} \ m < -c - \frac{2\mu c}{M+2\mu} \end{cases}, \tag{13}$$

and

$$\pi_B(m) = \begin{cases} 1 \ \mathit{if} \ m \geq c - \frac{2\mu c}{M+2\mu} \\ 0 \ \mathit{if} \ m < c - \frac{2\mu c}{M+2\mu} \end{cases}. \tag{14}$$

A.1: The incentive compatibility constraints

The first step is to simplify the incentive compatibility constraints. Consider first the constraints for A households. Clearly, if $s_A + m$ exceeds $s_B - c$, so that the household prefers that their child is assigned to school A, then the household will want to report an m' that maximizes $\pi_A(m')$ - the probability they are assigned to school A. Thus, incentive compatibility requires that $\pi_A(m)$ equal $\max \pi_A(m')$. Similarly, if $s_A + m$ is less than $s_B - c$, so that the household prefers that their child is assigned to school B, then the household will want to report an m' that minimizes $\pi_A(m')$. Thus, incentive compatibility requires that $\pi_A(m)$ equal $\min \pi_A(m')$. A similar story holds for neighborhood B households. If $s_A + m - c$ exceeds s_B, then the household will want to report an m' that maximizes $\pi_B(m')$. Thus, incentive compatibility requires that $\pi_B(m)$ equal $\max \pi_B(m')$. On the other hand, if $s_A + m - c$ is less than s_B, then the household will want to report an m' that minimizes $\pi_B(m')$. Thus, incentive compatibility requires that $\pi_B(m)$ equal $\min \pi_B(m')$.

This discussion motivates:

Lemma 1 *An assignment $(\pi_A(m), \pi_B(m))$ is incentive compatible if and only if*

$$\pi_A(m) = \begin{cases} \overline{\pi}_A \text{ if } m \geq -c - \Delta s \\ \underline{\pi}_A \text{ if } m < -c - \Delta s \end{cases}, \tag{15}$$

and

$$\pi_B(m) = \begin{cases} \overline{\pi}_B \text{ if } m \geq c - \Delta s \\ \underline{\pi}_B \text{ if } m < c - \Delta s \end{cases}, \tag{16}$$

where $\overline{\pi}_J = \max \pi_J(m')$ and $\underline{\pi}_J = \min \pi_J(m')$ for $J \in \{A, B\}$ and Δs solves the equation

$$\Delta s = 2\mu[\Pr(m > -c - \Delta s)\,\overline{\pi}_A + \Pr(m < -c - \Delta s)\,\underline{\pi}_A \\ - \Pr(m > c - \Delta s)\,\overline{\pi}_B - \Pr(m < c - \Delta s)\,\underline{\pi}_B]. \tag{17}$$

It follows from Lemma 1 that incentive compatible assignments can be characterized by four probabilities $(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$, which must satisfy $0 \leq \underline{\pi}_J \leq \overline{\pi}_J \leq 1$, along with an associated peer quality difference Δs, which has to satisfy (17). The associated assignment is then given by (15) and (16). This allows us to describe incentive compatible assignments with the notation $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$.[1] Intuitively, the probability $\overline{\pi}_J$ is the probability that households from neighborhood J who want their children to attend school A are assigned to school A and $\underline{\pi}_J$ is the probability that households from neighborhood J who want their children to attend school B are assigned to school A.

A.2: Two simplifying observations

Our first observation about second best assignments concerns the $\underline{\pi}_J$'s. Intuitively, it would not seem sensible for the school district to assign neighborhood J children to school A whose parents do not want them to attend school A if it were simultaneously denying admission to neighborhood J children whose parents do want their children to attend school A. After all, it would be better to simply substitute those who did want to attend for those who did not. This logic is confirmed by the following Lemma:

Lemma 2 *Suppose that $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ is a second best assignment. Then, if $\overline{\pi}_A < 1$ and $M > \max\{c + \Delta s, -c - \Delta s\}$, it must be the case that $\underline{\pi}_A = 0$. Moreover, if $\overline{\pi}_B < 1$ and $M > \max\{c - \Delta s, -c + \Delta s\}$, it must be the case that $\underline{\pi}_B = 0$.*

The caveat that M exceeds $\max\{c + \Delta s, -c - \Delta s\}$ just ensures that there are some A households who do and do not want their children to attend school A. If this is not the case, then either $\overline{\pi}_A$ or $\underline{\pi}_A$ is irrelevant for the school district's payoff. Similarly, the requirement that M exceeds $\max\{c - \Delta s, -c + \Delta s\}$ ensures that there are some B households who do and some who do not want their children to attend school A.

Our second observation concerns the peer quality difference. Intuitively, it seems that this must be non-negative for a second best assignment. Creating a negative peer quality

[1] To the extent that equation (17) uniquely defines Δs, given $(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$, then it is only necessary to specify $(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$ to characterize an assignment. While it is the case that there is a unique Δs corresponding to $(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$ for the assignments we consider, we include Δs for clarity.

difference would require sending more than 50% of A households to school B, which would seem to create a cross-hauling inefficiency. This conjecture is confirmed in:

Lemma 3 *Suppose that* $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ *is a second best assignment. Then,* $\Delta s \geq 0$.

To prove this result we show that any feasible, incentive compatible assignment $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ which is such that Δs is negative can be dominated by either neighborhood assignment or what we call the *zero peer quality difference assignment.* The latter assignment involves assigning A households who wish to attend school B given a zero peer quality difference to school B with probability 1; assigning A households who wish to attend school A to school A with probability $M/(M+c)$; assigning B households who wish to attend school A to school A with probability 1; and assigning B households who do not wish to attend school A to school A with probability $c/(M+c)$. Crucially, this assignment forces some B households to attend school A and some A households to attend school B even when they do not want to. In this way, it achieves an equal mix of A and B households in both schools and thereby eliminates the peer quality difference between the two schools.

An important implication of Lemma 3 is that in a second best assignment there are always some neighborhood B households who would like their children to attend school A. This follows since $M + \Delta s$ must exceed c. The consequence of this is that school A will be the school in excess demand.

A.3: The $\underline{\pi}_J$'s

Lemma 2 tells us that $\underline{\pi}_J$ equals 0 if $\overline{\pi}_J$ is less than 1, but what about if $\overline{\pi}_J$ equals 1? Intuitively, it seems that $\underline{\pi}_A$ should equal 0 even if $\overline{\pi}_A$ is equal to 1. Assigning neighborhood A children to school A whose parents do not want them to attend school A would not seem to make sense when the nature of the incentive problem is that parents will be too eager to send their children to school A because of the peer quality difference. Surely it would be better to replace such children with B children whose parents do want their children to attend school A? The following result confirms this logic.

Lemma 4 *Suppose that* $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ *is a second best assignment such that* $M > c + \Delta s$. *Then,* $\underline{\pi}_A = 0$.

The caveat that M must exceed $c + \Delta s$, reflects the fact that if all A households wish their students to attend school A (which would be the case if M is less than $c + \Delta s$) then it does not matter what $\underline{\pi}_A$ is. To prove Lemma 4 we start with a feasible, incentive compatible assignment $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ such that Δs is non-negative and $\underline{\pi}_A$ is positive. We then show that, if some neighborhood A households wish their children to attend school B, marginally decreasing $\underline{\pi}_A$ and compensating by increasing $\overline{\pi}_B$ must increase the school district's payoff. The analysis takes into account that such a change will impact the peer quality difference Δs.

It is much less obvious that $\underline{\pi}_B$ must equal 0 when $\overline{\pi}_B$ equals 1. Intuitively, it seems possible that assigning B children to school A whose parents do not want them to attend school A may be justified on the grounds of reducing the peer quality difference. Indeed, this was the logic behind the zero peer quality difference assignment and this assignment yields a higher payoff for the school district than neighborhood assignment under some conditions.[2] It turns out, however, that it is never optimal to force neighborhood B households to attend school A.

Lemma 5 *Suppose that* $\{(\overline{\pi}_A, \underline{\pi}_A, 1, \underline{\pi}_B), \Delta s\}$ *is a second best assignment such that* $M > -c + \Delta s$*. Then,* $\underline{\pi}_B = 0$.

The caveat that M exceeds $-c + \Delta s$ ensures that there are some neighborhood B households who wish their children to go to school B. If that is not the case, $\underline{\pi}_B$ is irrelevant. Lemma 5 is proved by showing that if we start with a feasible, incentive compatible assignment $\{(\overline{\pi}_A, 0, 1, \underline{\pi}_B), \Delta s\}$ such that Δs is non-negative, $\underline{\pi}_B$ is positive, and some B households wish their children to go to school B, we can always increase the school district's payoff by reducing $\underline{\pi}_B$ to zero and raising $\overline{\pi}_A$ to keep school A full. This is established by keeping track of all the impacted households and showing that the gains of the households who benefit from the change exceed the losses of the households who lose.

A.4: The $\overline{\pi}_J$'s

We now turn our attention to the $\overline{\pi}_J$'s. Our main result is that the second best assignment *either* admits all A children whose parents want them to attend school A *or* admits all B children whose parents want them to attend school A. The most important policy implication of this is that it rules out the possibility that seats in school A are randomly allocated among interested households.

Lemma 6 *Assume that* $c \neq 2\mu$ *and suppose that* $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ *is a second best assignment. Then, either* $\overline{\pi}_A = 1$ *or* $\overline{\pi}_B = 1$.

To prove this result we show that if it is not the case that either $\overline{\pi}_A$ or $\overline{\pi}_B$ equals 1, it is always possible to change the assignment in such a way as to create a gain in the school district's payoff. The nature of the change depends on the assignment that we start with. In particular, it matters whether or not in the original assignment any neighborhood A households wish their children to attend school B. If all A households wish their children to attend school A in the original assignment, it can be dominated by neighborhood assignment. If some A households wish their children to attend school B, the original assignment can be dominated by an assignment in which $\overline{\pi}_B$ is marginally decreased if c exceeds 2μ and increased if c is less than 2μ.

A.5: A priority assignments

Having established that the second best assignment involves either admitting all A children whose parents want them to attend school A or admitting all B children whose parents want them to attend school A, we next turn to understanding the other features of these assignments. This sub-section considers assignments in which $\overline{\pi}_A$ equals 1.

[2] The condition is that M exceed $3c$.

Lemma 7 *Suppose that* $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ *is a second best assignment. Then, if* $M \leq c + 2\mu$, $\underline{\pi}_A \in [0, 1]$ *and* $(\overline{\pi}_B, \underline{\pi}_B, \Delta s) = (0, 0, 2\mu)$, *while if* $M > c + 2\mu$

$$(\underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B, \Delta s) = \left(0, \frac{M - c - \frac{2\mu c}{M-2\mu}}{M - c + \frac{2\mu c}{M-2\mu}}, 0, \frac{2\mu c}{M - 2\mu}\right). \tag{18}$$

This result reflects the fact that, if M is less than $c + 2\mu$, all A households wish to attend school A. Since A households are given priority, there is no room for interested B households. This implies that both $\overline{\pi}_B$ and $\underline{\pi}_B$ must equal 0. It also means that the peer quality difference is 2μ. It does not matter what $\underline{\pi}_A$ is in this case, because all A households wish to attend school A. If M exceeds $c + 2\mu$, then, some A households wish to attend school B which creates some seats for interested B households. The peer quality difference in this case is $2\mu c/(M - 2\mu)$, which is smaller than 2μ because of the B children in school A and A children in school B. In this case, $\underline{\pi}_A$ is equal to 0 by Lemma 4 and $\underline{\pi}_B$ is equal to 0 by Lemma 2.

We will describe an assignment $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ such that $(\underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$ and Δs satisfy the conditions of Lemma 7 as an A *priority assignment*. Using (6), we can develop a simple expression for the school district's payoff under an A priority assignment.

Lemma 8 *Suppose that* $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ *is an A priority assignment. Then, the school district's payoff is*

$$W_A = \begin{cases} 0 \text{ if } M \leq c + 2\mu \\ \frac{(M-c)^2 - (M-c)\Delta s_A}{2M} \text{ if } M > c + 2\mu \end{cases}, \tag{19}$$

where

$$\Delta s_A \equiv \frac{2\mu c}{M - 2\mu}.$$

A.6: B Priority Assignments

Turning to assignments in which $\overline{\pi}_B$ equals 1, we have:

Lemma 9 *Suppose that* $\{(\overline{\pi}_A, \underline{\pi}_A, 1, \underline{\pi}_B), \Delta s\}$ *is a second best assignment. Then, if* $M \leq \frac{c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}}{2}$, $\underline{\pi}_A \in [0, \overline{\pi}_A]$ *and*

$$(\overline{\pi}_A, \underline{\pi}_B, \Delta s) = \left(\frac{M + c - \frac{2\mu c}{M+2\mu}}{2M}, 0, \frac{2\mu c}{M + 2\mu}\right), \tag{20}$$

while, if $M > \frac{c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}}{2}$,

$$(\overline{\pi}_A, \underline{\pi}_A, \underline{\pi}_B, \Delta s) = \left(\frac{M + c - \frac{2\mu c}{M+2\mu}}{M + c + \frac{2\mu c}{M+2\mu}}, 0, 0, \frac{2\mu c}{M + 2\mu}\right). \tag{21}$$

If M is less than $\left(c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}\right)/2$ then some B households wish to attend school A but no A households wish to attend school B. B priority means that interested B households are assigned to school A which forces $\overline{\pi}_A$ below 1. Lemma 5 implies that $\underline{\pi}_B$

equals 0. It does not matter what $\underline{\pi}_A$ is in this case, as long as it is less than $\overline{\pi}_A$, because all A households wish to attend school A. The peer quality difference is $2\mu c/(M+2\mu)$ which is smaller than 2μ because of the mixing of children across schools.

If M exceeds $\left(c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}\right)/2$, then some A households wish to attend school B. Because of B priority and the fact that the fraction of neighborhood B switchers exceeds the fraction of neighborhood A switchers, $\overline{\pi}_A$ is less than 1. Lemma 2 implies that $\underline{\pi}_A$ equals 0 and Lemma 5 implies that $\underline{\pi}_B$ equals 0. The peer quality difference is again $2\mu c/(M+2\mu)$. The fact that this is the same as in the previous case reflects the fact that, under B priority, the fraction of B households in school A is determined by the preferences of neighborhood B households.

We will describe an assignment $\{(\overline{\pi}_A,\underline{\pi}_A,1,\underline{\pi}_B),\Delta s\}$ such that $(\overline{\pi}_A,\underline{\pi}_A,\underline{\pi}_B)$ and Δs satisfy the conditions of Lemma 9 as a B *priority assignment*. We can use Lemma 9 and (6), to develop a simple expression for the school district's payoff under a B priority assignment.

Lemma 10 *Suppose that* $\{(\overline{\pi}_A,\underline{\pi}_A,1,0),\Delta s\}$ *is a* B *priority assignment. Then, the school district's payoff is*

$$W_B=\begin{cases}\left(\frac{M+\Delta s_B-c}{2M}\right)\left(\frac{M-\Delta s_B-c}{2M}-c\right) & \text{if } M\le\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2}\\ \frac{(M-c)^2-(M+c)\Delta s_B}{2M} & \text{if } M>\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2}\end{cases},\tag{22}$$

where

$$\Delta s_B=\frac{2\mu c}{M+2\mu}.$$

A.7: A vs B priority assignments

The next task is to compare the school district's payoff under A and B priority assignments using the expressions presented in Lemma 8 and 10. This yields:

Lemma 11 *i) If* $c>2\mu$, *the school district's payoff is higher under an* A *priority assignment. ii) If* $c<2\mu$, *the school district's payoff is higher under an* A *priority assignment if* $M<c-\mu+\sqrt{\mu^2+4c\mu}$, *and higher under a* B *priority assignment if* $M>c-\mu+\sqrt{\mu^2+4c\mu}$.

A.8: Establishing Proposition 3

We are now ready to establish Proposition 3. Suppose first that c exceeds 2μ. Lemma 6 along with Lemma 7 and 9 imply that if $\{(\overline{\pi}_A,\underline{\pi}_A,\overline{\pi}_B,\underline{\pi}_B),\Delta s\}$ is a second best assignment it is either an A priority assignment or a B priority assignment. By Lemma 11, the school district's payoff is higher under an A priority assignment. Thus, the second best assignment must be an A priority assignment.

From Lemma 7, we know that if M is less than $c+2\mu$, an A priority assignment has $\underline{\pi}_A\in[0,1]$ and $(\overline{\pi}_A,\overline{\pi}_B,\underline{\pi}_B,\Delta s)$ equal to $(1,0,0,2\mu)$. As we noted, it does not matter what $\underline{\pi}_A$ is, because all A households want to attend school A. It follows that, if M is less than $c+2\mu$, the second best assignment is $(\pi_A(m),\pi_B(m))=(1,0)$ for all m. If M exceeds $c+2\mu$, Lemma 7 tells us that an A priority assignment is such that

$$\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\} = \left\{\left(1, 0, \frac{M - c - \frac{2\mu c}{M-2\mu}}{M - c + \frac{2\mu c}{M-2\mu}}, 0\right), \frac{2\mu c}{M - 2\mu}\right\}.$$

Accordingly, from Lemma 1, the second best assignment when M exceeds $c + 2\mu$ is

$$\pi_A(m) = \begin{cases} 1 \text{ if } m \geq -c - \frac{2\mu c}{M-2\mu} \\ 0 \text{ if } m < -c - \frac{2\mu c}{M-2\mu} \end{cases},$$

and

$$\pi_B(m) = \begin{cases} \frac{M-c-\frac{2\mu c}{M-2\mu}}{M-c+\frac{2\mu c}{M-2\mu}} \text{ if } m \geq c - \frac{2\mu c}{M-2\mu} \\ 0 \text{ if } m < c - \frac{2\mu c}{M-2\mu} \end{cases}.$$

This confirms part i) of the proposition.

Now suppose that c is less than 2μ. Again, Lemma 6 along with Lemma 7 and 9 imply that if $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ is a second best assignment it is either an A priority assignment or a B priority assignment. By Lemma 11, if M is less than $c-\mu+\sqrt{\mu^2 + 4c\mu}$, the school district's payoff is higher under an A priority assignment, while if M exceeds $c - \mu + \sqrt{\mu^2 + 4c\mu}$, the school district's payoff is higher under a B priority assignment.

Note that if c is less than 2μ, we have that $\sqrt{\mu^2 + 4c\mu} - \mu$ is less than 2μ and thus if M is less than $c - \mu + \sqrt{\mu^2 + 4c\mu}$ it is also true that M is less than $c + 2\mu$. From Lemma 7, we know that if M is less than $c + 2\mu$, an A priority assignment has $\underline{\pi}_A \in [0, 1]$ and $(\overline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B, \Delta s)$ equal to $(1, 0, 0, 2\mu)$. As just noted, it does not matter what $\underline{\pi}_A$ is, because all A households want to attend school A. It follows that, if M is less than $c-\mu+\sqrt{\mu^2 + 4c\mu}$, the second best assignment is $(\pi_A(m), \pi_B(m)) = (1, 0)$ for all m.

Note also that if c is less than 2μ, we have that

$$c - \mu + \sqrt{\mu^2 + 4c\mu} > \frac{c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}}{2}.$$

Moreover, if M exceeds $\left(c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}\right)/2$, Lemma 9 tells us that a B priority assignment is such that

$$\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\} = \left\{\left(\frac{M + c - \frac{2\mu c}{M+2\mu}}{M + c + \frac{2\mu c}{M+2\mu}}, 0, 1, 0\right), \frac{2\mu c}{M + 2\mu}\right\}.$$

Accordingly, from Lemma 1, the second best assignment when M exceeds $c-\mu+\sqrt{\mu^2 + 4c\mu}$ is

$$\pi_A(m) = \begin{cases} \frac{M+c-\frac{2\mu c}{M+2\mu}}{M+c+\frac{2\mu c}{M+2\mu}} \; if \; m \geq -c - \frac{2\mu c}{M+2\mu} \\ 0 \; if \; m < -c - \frac{2\mu c}{M+2\mu} \end{cases},$$

and

$$\pi_B(m) = \begin{cases} 1 \; if \; m \geq c - \frac{2\mu c}{M+2\mu} \\ 0 \; if \; m < c - \frac{2\mu c}{M+2\mu} \end{cases}.$$

This confirms part ii) of the Proposition.

Appendix B. Proof of Proposition 1 and Lemmas 1-11

B.1: Derivation of (6)

The sum of households' expected payoffs is

$$\begin{array}{c}\int_{-M}^{M}\left(\pi_A(m)\left(s_A+m\right)+\left(1-\pi_A(m)\right)\left(s_B-c\right)\right)\frac{dm}{2M}\\ +\int_{-M}^{M}\left(\pi_B(m)\left(s_A+m-c\right)+\left(1-\pi_B(m)\right)s_B\right)\frac{dm}{2M}\end{array}.$$

After some rearrangement, this can be written as:

$$\begin{array}{c}\left(s_A-s_B\right)\left(\int_{-M}^{M}\pi_A(m)\frac{dm}{2M}+\int_{-M}^{M}\pi_B(m)\frac{dm}{2M}\right)+2s_B\\ +\int_{-M}^{M}\left[m\pi_A(m)+m\pi_B(m)-c\left(1-\pi_A(m)\right)-c\pi_B(m)\right]\frac{dm}{2M}.\end{array}$$

Using (2) and (3), we can show that

$$\left(s_A-s_B\right)\left(\int_{-M}^{M}\pi_A(m)\frac{dm}{2M}+\int_{-M}^{M}\pi_B(m)\frac{dm}{2M}\right)+2s_B=0.$$

B.2: Proof of Proposition 1

First consider the problem of choosing an assignment $(\pi_A(m), \pi_B(m))$ to maximize the objective function (6) ignoring the feasibility constraint. The first order conditions for $\pi_A(m)$ and $\pi_B(m)$ imply that: if $m+c>0$, then $\pi_A(m)=1$; if $m+c<0$, then $\pi_A(m)=0$; if $m-c>0$, then $\pi_B(m)=1$; and if $m-c<0$, then $\pi_B(m)=0$.

Now observe that these rules satisfy the feasibility constraint. If $M>c$, then they imply that

$$\begin{aligned}\int_{-M}^{M}\left(\pi_A(m)+\pi_B(m)\right)\frac{dm}{2M} &= \int_{-c}^{M}\frac{dm}{2M}+\int_{c}^{M}\frac{dm}{2M}\\ &= \frac{M+c}{2M}+\frac{M-c}{2M}=1.\end{aligned}$$

If $M<c$, then they imply that

$$\int_{-M}^{M}\left(\pi_A(m)+\pi_B(m)\right)\frac{dm}{2M}=\int_{-M}^{M}\frac{dm}{2M}=\frac{2M}{2M}=1.$$

The unconstrained optimal rules must therefore solve the constrained problem.

B.3: Proof of Lemma 1

Suppose that the assignment $(\pi_A(m), \pi_B(m))$ is incentive compatible. Then (15) and (16) hold as explained in the paragraph preceding the statement of Lemma 1. Given (15) and (16), then (2) and (3) imply that

$$\begin{array}{c}s_A=\mu\left[\Pr(m>-c-\Delta s)\overline{\pi}_A+\left(1-\Pr m>-c-\Delta s\right)\right)\underline{\pi}_A]\\ -\mu\left[\Pr\left(m>c-\Delta s\right)\overline{\pi}_B+\left(1-\Pr\left(m>c-\Delta s\right)\right)\underline{\pi}_B\right]\end{array}$$

and

$$\begin{aligned} s_B = &\mu\left[\Pr\left(m > -c-\Delta s\right)\left(1-\overline{\pi}_A\right) + \left(1-\Pr\left(m > -c-\Delta s\right)\right)\left(1-\underline{\pi}_A\right)\right] \\ &-\mu\left[\Pr\left(m > c-\Delta s\right)\left(1-\overline{\pi}_B\right) + \left(1-\Pr\left(m > c-\Delta s\right)\right)\left(1-\underline{\pi}_B\right)\right]. \end{aligned}$$

Thus, we have that

$$\Delta s = s_A - s_B = 2\mu\left[\begin{array}{c} \Pr\left(m > -c-\Delta s\right)\overline{\pi}_A + \left(1-\Pr\left(m > -c-\Delta s\right)\right)\underline{\pi}_A \\ -\Pr(m > c-\Delta s)\overline{\pi}_B - \left(1-\Pr\left(m > c-\Delta s\right)\right)\underline{\pi}_B \end{array}\right],$$

which is (17).

Conversely, suppose that there exist numbers $(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B)$ such that $0 \leq \underline{\pi}_J \leq \overline{\pi}_J \leq 1$ and that Δs satisfies (17). Then the assignment defined by (15) and (16) satisfies the constraints (9) and (10) when s_A and s_B are given by

$$\begin{aligned} s_A = &\mu\left[\Pr\left(m > -c-\Delta s\right)\overline{\pi}_A + \left(1-\Pr\left(m > -c-\Delta s\right)\right)\underline{\pi}_A\right] \\ &-\mu\left[\Pr\left(m > c-\Delta s\right)\right)\overline{\pi}_B + \left(1-\Pr\left(m > c-\Delta s\right)\right)\underline{\pi}_B] \end{aligned}$$

and

$$\begin{aligned} s_B = &\mu\left[\Pr\left(m > -c-\Delta s\right)\left(1-\overline{\pi}_A\right) + \left(1-\Pr\left(m > -c-\Delta s\right)\right)\left(1-\underline{\pi}_A\right)\right] \\ &-\mu\left[\Pr\left(m > c-\Delta s\right)\right)\left(1-\overline{\pi}_B\right) + \left(1-\Pr\left(m > c-\Delta s\right)\right)\left(1-\underline{\pi}_B\right)]. \end{aligned}$$

The assignment defined by (15) and (16) is therefore incentive compatible.

B.4: Proof of Lemma 2

Suppose $\overline{\pi}_A < 1$, $M > c + \Delta s$, and $\underline{\pi}_A > 0$. Consider increasing $\overline{\pi}_A$ by $\varepsilon / \Pr\left(m > -c-\Delta s\right)$ and decreasing $\underline{\pi}_A$ by $\varepsilon / \left(1-\Pr\left(m > -c-\Delta s\right)\right)$ where ε is small enough to prevent violation of boundary conditions. Note that the assumption that $M > \max\left\{c+\Delta s, -c-\Delta s\right\}$ implies that $1 > \Pr\left(m > -c-\Delta s\right) > 0$. This change has no impact on Δs and hence the household types choosing the various options. The feasibility constraint remains satisfied. The change in the school district's payoff is

$$\left[E(m|\, m > -c-\Delta s) - E(m|\, m < -c-\Delta s)\right]\varepsilon,$$

which is positive. Intuitively, the change reduces the assignment of A households who do not want their children to attend school A and replaces them with an equal number of A households who do want their children to attend school A. This has no impact on feasibility or peer quality, but does increase match benefits.

A similar argument applies for B. Suppose $\overline{\pi}_B < 1$ and $\underline{\pi}_B > 0$. Consider increasing $\overline{\pi}_B$ by $\varepsilon / \Pr\left(m > c-\Delta s\right))$ and decreasing $\underline{\pi}_B$ by $\varepsilon / \left(1-\Pr\left(m > c-\Delta s\right)\right)$. The assumption that $M > \max\left\{c-\Delta s, -c+\Delta s\right\}$ implies that $1 > \Pr\left(m > c-\Delta s\right) > 0$. This change has no impact on Δs and hence the household types choosing the various option. The feasibility constraint remains satisfied. The change in the objective function is

$$\left[E(m|\, m > c-\Delta s) - E(m|\, m < c-\Delta s)\right]\varepsilon,$$

which is positive.

B.5: Proof of Lemma 3

Consider a feasible, incentive compatible assignment $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ which is such that $\Delta s < 0$. We will show that this assignment can be dominated either by neighborhood assignment or by the zero peer quality difference assignment that is discussed in Appendix A following the statement of Lemma 3.

It is helpful to first calculate the school district's payoffs for our two benchmark assignments. Neighborhood assignment obviously generates a payoff of 0. To understand the payoff from the zero peer quality difference assignment, consider Figure C1 which illustrates how the different household types are assigned to the two schools. There are four groups of households: A households who prefer school B given a zero peer quality difference; A households who prefer school A; B households who prefer school B; and B households who prefer school A. The first and fourth groups get their preferred school; the second and third groups get their preferred school with probabilities $M/(M+c)$ and $1-c/(M+c)$ respectively. Summing up the payoffs generated by the four groups, we get

$$\frac{M-c}{2M}(-c)+\frac{M+c}{2M}\left((\frac{M}{M+c})\frac{M-c}{2}+(1-\frac{M}{M+c})(-c)\right)$$
$$+\frac{M+c}{2M}\left((\frac{c}{M+c})(\frac{c-M}{2}-c)+(1-\frac{c}{M+c})(0)\right)+\frac{M-c}{2M}(\frac{M+c}{2}-c).$$

This reduces to

$$\frac{M-3c}{2}.$$

Note that this assignment dominates neighborhood assignment when $M > 3c$.

We now turn to the negative Δs assignment. In this assignment, it must be the case that more than 1/2 of the households living in neighborhood A attend school B and more than 1/2 of the households living in neighborhood B attend school A. We may assume without loss of generality that $\overline{\pi}_B = 1$. To see this suppose that $\overline{\pi}_B < 1$. Then, by Lemma 2, if $M > c - \Delta s$, it must be the case that $\underline{\pi}_B = 0$. But, since $\Delta s < 0$ school B has the peer advantage and it cannot be that more than 1/2 of neighborhood B households wish to attend school A. Thus, the fraction of neighborhood B households attending school A is less than 1/2, which contradicts the fact that $\Delta s < 0$. If $M \leq c - \Delta s$, then $\overline{\pi}_B$ is irrelevant since neighborhood B households wish to attend school B. Thus, it can be set equal to 1.

There are a number of possibilities to consider. There are two main cases, each of which has three sub-cases. The two main cases are $\underline{\pi}_A = 0$ and $\underline{\pi}_A > 0$. In each case, the three sub-cases are i) $-\Delta s < M - c$, ii) $M - c \leq -\Delta s < M + c$; and iii) $M + c \leq -\Delta s$. We tackle each in turn.

B.5.1: $\underline{\pi}_A = 0$

i) $-\Delta s < M - c$ This situation is depicted in Figure C2. There are again four groups of households. Summing up the payoffs generated by the four groups, we get

$$\frac{M-c-\Delta s}{2M}(-c) + \frac{M+c+\Delta s}{2M}\left(\overline{\pi}_A(\frac{M-c-\Delta s}{2}) + (1-\overline{\pi}_A)(-c)\right)$$
$$+\frac{M+c-\Delta s}{2M}\left(\underline{\pi}_B(\frac{c-\Delta s-M}{2}-c) + (1-\underline{\pi}_B)(0)\right) + \frac{M-c+\Delta s}{2M}(\frac{M+c-\Delta s}{2}-c).$$

This can be rearranged to yield

$$-c+\frac{M+c+\Delta s}{2M}\overline{\pi}_A(\frac{M+c-\Delta s}{2})+\frac{M+c-\Delta s}{2M}\underline{\pi}_B(\frac{-c-\Delta s-M}{2})+\frac{M-c+\Delta s}{2M}(\frac{M-c-\Delta s}{2}).$$

We will further simplify this expression by using what we know about the determinants of the peer difference.

The fraction of A households attending school A is

$$\frac{M+c+\Delta s}{2M}\overline{\pi}_A$$

and thus, since the assignment is feasible, the fraction of B households attending school A must be

$$1-\frac{M+c+\Delta s}{2M}\overline{\pi}_A.$$

Accordingly, we have that

$$\begin{aligned}\Delta s &= 2\mu\left(\frac{M+c+\Delta s}{2M}\overline{\pi}_A - \left(1-\frac{M+c+\Delta s}{2M}\overline{\pi}_A\right)\right)\\ &= 2\mu\left(\frac{M+c+\Delta s}{M}\overline{\pi}_A - 1\right)\end{aligned}$$

It follows that

$$\frac{M+c+\Delta s}{2M}\overline{\pi}_A = \frac{\Delta s+2\mu}{4\mu}. \tag{23}$$

Similarly, the fraction of B households attending school A is

$$\frac{M+c-\Delta s}{2M}\underline{\pi}_B + \frac{M-c+\Delta s}{2M}$$

and thus, since the assignment is feasible, the fraction of A households attending school A must be

$$1-\frac{M+c-\Delta s}{2M}\underline{\pi}_B - \frac{M-c+\Delta s}{2M}.$$

Accordingly, we have that

$$\begin{aligned}\Delta s &= 2\mu\left(1-\frac{M+c-\Delta s}{2M}\underline{\pi}_B - \frac{M-c+\Delta s}{2M} - \left(\frac{M+c-\Delta s}{2M}\underline{\pi}_B + \frac{M-c+\Delta s}{2M}\right)\right)\\ &= 2\mu\left(1-\frac{M+c-\Delta s}{M}\underline{\pi}_B - \frac{M-c+\Delta s}{M}\right).\end{aligned}$$

It follows that

$$\frac{M+c-\Delta s}{2M}\underline{\pi}_B = \frac{2\mu c - \Delta s\,(2\mu+M)}{4\mu M}. \tag{24}$$

Substituting (23) and (24) into the school district's objective function yields

$$-c+(\frac{\Delta s+2\mu}{4\mu})(\frac{M+c-\Delta s}{2})+(\frac{2\mu c-\Delta s\,(2\mu+M)}{4\mu M})(\frac{-c-\Delta s-M}{2})+\frac{(M-c)^2-\Delta s^2}{4M}. \tag{25}$$

Note that this payoff depends only on Δs.

Now consider the thought experiment of choosing a different level of Δs. Observe that setting $\Delta s = 0$, would yield a payoff of $(M-3c)/2$ which is the payoff generated by the zero peer quality difference assignment. Differentiating (25) with respect to Δs, we obtain

$$\frac{M+c-2\Delta s-2\mu}{8\mu}+\frac{2\Delta s\,(2\mu+M)+M\,(M+2\mu+c)}{8\mu M}-\frac{\Delta s}{2M}.$$

After some cancellations and rearranging, this reduces to

$$\frac{M+c}{4\mu},$$

which is positive. Thus, the payoff generated by the negative Δs assignment, is less than that generated by the zero peer quality difference assignment.

ii) $M-c \leq -\Delta s < M+c$ This situation is depicted in Figure C3. There are now just three groups of households because all B households wish to attend school B. Summing up the payoffs generated by the three groups, we get

$$\frac{M-c-\Delta s}{2M}(-c)+\frac{M+c+\Delta s}{2M}\left(\overline{\pi}_A(\frac{M-c-\Delta s}{2})+(1-\overline{\pi}_A)(-c)\right)+\underline{\pi}_B(-c)+(1-\underline{\pi}_B)(0).$$

This simplifies to

$$-c+\frac{M+c+\Delta s}{2M}\overline{\pi}_A(\frac{M+c-\Delta s}{2})-\underline{\pi}_B c,$$

which equals

$$-c+\frac{(M+c)^2-\Delta s^2}{4M}\overline{\pi}_A-\underline{\pi}_B c.$$

We know that the fraction of B households attending school A is $\underline{\pi}_B$ and, thus, since the assignment is feasible, the fraction of A households attending school A must be $1-\underline{\pi}_B$. Accordingly, we have that $\Delta s = 2\mu(1-2\underline{\pi}_B)$.

The fact that $\Delta s < 0$, implies that $\underline{\pi}_B > 1/2$. Moreover, since $\overline{\pi}_A \in [0,1]$ and $\Delta s^2 \geq (M-c)^2$, we have that the school district's payoff is less than

$$-\frac{3}{2}c+\frac{(M+c)^2-(M-c)^2}{4M}.$$

This equals

$$-\frac{3}{2}c+\frac{4cM}{4M}=-\frac{c}{2}.$$

This payoff is negative and hence the negative Δs assignment generates a lower payoff than does neighborhood assignment.

iii) $M + c \leq -\Delta s$ In this case, there are now just two groups of households because all households wish to attend school B. Given that $\underline{\pi}_A = 0$, all A households are assigned to school B. Feasibility demands that $\underline{\pi}_B = 1$, so all B households are assigned to school A. This policy is obviously dominated by neighborhood assignment.

B.5.2: $\underline{\pi}_A > 0$

i) $-\Delta s < M - c$ In this case, Lemma 2 implies that $\overline{\pi}_A = 1$. This situation is depicted in Figure C4. There are four groups of households. Summing up the payoffs generated by the four groups, we get

$$\frac{M-c-\Delta s}{2M}\left(\underline{\pi}_A(\frac{-c-\Delta s-M}{2}) + (1-\underline{\pi}_A)(-c)\right) + \frac{M+c+\Delta s}{2M}(\frac{M-c-\Delta s}{2})$$
$$+\frac{M+c-\Delta s}{2M}\left(\underline{\pi}_B(\frac{c-\Delta s-M}{2} - c) + (1-\underline{\pi}_B)(0)\right) + \frac{M-c+\Delta s}{2M}(\frac{M+c-\Delta s}{2} - c).$$

This can be rearranged to yield

$$\frac{M-c-\Delta s}{2M}\underline{\pi}_A(\frac{c-\Delta s-M}{2}) + \frac{M-c+\Delta s}{2M}(\frac{M-c-\Delta s}{2})$$
$$+\frac{M+c-\Delta s}{2M}\underline{\pi}_B(\frac{-c-\Delta s-M}{2}) + \frac{M-c+\Delta s}{2M}(\frac{M-c-\Delta s}{2}).$$

The fraction of A households attending school A is

$$\frac{M+c+\Delta s}{2M} + \frac{M-c-\Delta s}{2M}\underline{\pi}_A$$

and thus, since the assignment is feasible, the fraction of B households attending school A must be

$$1 - \frac{M+c+\Delta s}{2M} - \frac{M-c-\Delta s}{2M}\underline{\pi}_A.$$

Accordingly, we have that

$$\Delta s = 2\mu\left(\frac{M+c+\Delta s}{M} + \frac{M-c-\Delta s}{M}\underline{\pi}_A - 1\right).$$

It follows that

$$\frac{M-c-\Delta s}{2M}\underline{\pi}_A = \frac{M\Delta s - (c+\Delta s)2\mu}{4\mu M}. \tag{26}$$

Similarly, the fraction of B households attending school A is

$$\frac{M+c-\Delta s}{2M}\underline{\pi}_B + \frac{M-c+\Delta s}{2M}$$

and thus, since the assignment is feasible, the fraction of A households attending school A must be

$$1 - \frac{M+c-\Delta s}{2M}\underline{\pi}_B - \frac{M-c+\Delta s}{2M}.$$

Accordingly, we have that

$$\Delta s = 2\mu\left(1 - \frac{M+c-\Delta s}{M}\underline{\pi}_B - \frac{M-c+\Delta s}{M}\right).$$

It follows that

$$\frac{M+c-\Delta s}{2M}\underline{\pi}_B = \frac{(c-\Delta s)2\mu - \Delta s M}{4\mu M}. \tag{27}$$

Substituting (26) and (27) into the school district's objective function yields

$$\frac{M\Delta s - (c+\Delta s)2\mu}{8\mu M}(c-\Delta s - M) + \frac{(M-c)^2 - \Delta s^2}{2M} + \frac{(c-\Delta s)2\mu - \Delta s M}{8\mu M}(-c-\Delta s - M).$$

Again, this payoff depends only on Δs. Rearranging and cancelling, we can write this as:

$$\frac{(M-c)^2 - c^2}{2M} + \frac{(2\mu+c)\,\Delta s}{4\mu}. \tag{28}$$

If this negative Δs assignment were optimal, the value of Δs would solve the problem of maximizing the above objective function, subject to the constraints that $\Delta s \in [-2\mu, 0]$ and the values of $\underline{\pi}_A$ and $\underline{\pi}_B$ implied by (26) and (27) are feasible in the sense of lying between 0 and 1. The constraint that $\underline{\pi}_B$ be feasible holds for all $\Delta s \in [-2\mu, 0]$ as does the constraint that $\underline{\pi}_A$ be less than 1. However, the constraint that $\underline{\pi}_A \geq 0$ requires that

$$(2\mu - M)(-\Delta s) \geq 2\mu c.$$

For there to exist any values of Δs in the interval $[-2\mu, 0]$ that satisfy this inequality requires that $2\mu > M + c$, so we may assume this to be true with no loss of generality.

Summarizing the argument so far, if our negative Δs assignment were optimal it would have to be the case that $2\mu > M + c$ and that the value of Δs would solve the problem of maximizing (28) subject to the constraint that $\Delta s \in [-2\mu, -2\mu c/(2\mu - M)]$. Now note that the derivative of (28) with respect to Δs is equal to $(2\mu + c)/4\mu$, which is positive. Thus, if our negative Δs assignment were optimal, it must be the case that $\Delta s = -2\mu c/(2\mu - M)$. With this optimal value of Δs, welfare is given by

$$\frac{(M-c)^2 - c^2}{2M} - \frac{(2\mu+c)\,c}{2\,(2\mu - M)}.$$

This equals

$$\frac{M}{2} - c - \frac{(2\mu+c)\,c}{2\,(2\mu - M)}.$$

This welfare level is strictly smaller than $(M - 3c)/2$ which is the welfare generated by the zero peer quality difference assignment.

ii) $M - c \leq -\Delta s < M + c$ In this case, Lemma 2 again implies that $\overline{\pi}_A = 1$. This situation is depicted in Figure C5. There are three groups of households because all B households wish to attend school B. Summing up the payoffs generated by the three groups, we get

$$\frac{M-c-\Delta s}{2M}\left(\underline{\pi}_A(\frac{-c-\Delta s-M}{2}) + (1-\underline{\pi}_A)(-c)\right) + \frac{M+c+\Delta s}{2M}(\frac{M-c-\Delta s}{2}) + \underline{\pi}_B(-c) + (1-\underline{\pi}_B)(0).$$

This simplifies to

$$\frac{M-c-\Delta s}{2M}\underline{\pi}_A(\frac{c-\Delta s-M}{2}) - \frac{M-c-\Delta s}{2M}c + \frac{M+c+\Delta s}{2M}(\frac{M-c-\Delta s}{2}) - c\underline{\pi}_B.$$

This equals

$$\frac{M-c-\Delta s}{2M}\underline{\pi}_A(\frac{c-\Delta s-M}{2}) + \frac{M-c+\Delta s}{2M}(\frac{M-c-\Delta s}{2}) - c\underline{\pi}_B,$$

which equals

$$\frac{(-\Delta s)^2 - (M-c)^2}{2M}\underline{\pi}_A + \frac{(M-c)^2 - \Delta s^2}{4M} - c\underline{\pi}_B.$$

This equals

$$(1-\underline{\pi}_A)\frac{(M-c)^2 - \Delta s^2}{4M} - c\underline{\pi}_B.$$

Since $M - c \leq -\Delta s$, this is negative and hence less than the payoff from neighborhood assignment.

iii) $M + c \leq -\Delta s$ In this case, there are now just two groups of households because all households wish to attend school B. Given that $\underline{\pi}_A > 0$, some A households are assigned to school B. Feasibility demands that $\underline{\pi}_B = \underline{\pi}_A$, so any equal number of B households are assigned to school A. Since households are randomly reassigned to schools outside of their neighborhood, this policy is dominated by neighborhood assignment.

B.6: Proof of Lemma 4

Suppose that $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ is a second best assignment such that $-M < -c - \Delta s$ and $\underline{\pi}_A > 0$. By Lemma 4, we know that $\Delta s \geq 0$. The fact that $-M < -c - \Delta s$ implies that $M > c + \Delta s \geq c - \Delta s$. The situation is illustrated in Figure C6. There are four groups of households: A households who prefer school B; A households who prefer school A; B households who prefer school B; and B households who prefer school A. The feasibility constraint implies that

$$\left(\frac{M-c-\Delta s}{2M}\right)\underline{\pi}_A + \frac{M+c+\Delta s}{2M} + \left(\frac{M+c-\Delta s}{2M}\right)\underline{\pi}_B + \left(\frac{M-c+\Delta s}{2M}\right)\overline{\pi}_B = 1.$$

Given that $\Delta s \geq 0$, $M > c + \Delta s$, and $\underline{\pi}_A > 0$, this inequality implies that $\overline{\pi}_B < 1$. It follows that $\underline{\pi}_B = 0$ by Lemma 2.

Given all this, the school district's payoff under this assignment is

$$\frac{M+c+\Delta s}{2M}(\frac{M-c-\Delta s}{2})+\frac{M-c-\Delta s}{2M}\left(\underline{\pi}_A(\frac{-M-c-\Delta s}{2})+(1-\underline{\pi}_A)(-c)\right)$$
$$+\frac{M-c+\Delta s}{2M}\underline{\pi}_B(\frac{M+c-\Delta s}{2}-c).$$

From the feasibility constraint, we have that

$$\frac{M-c+\Delta s}{2M}\underline{\pi}_B=1-\left(\frac{M-c-\Delta s}{2M}\right)\underline{\pi}_A-\frac{M+c+\Delta s}{2M}.$$

Substituting this into the school district's payoff, cancelling and simplifying, the payoff can be written as

$$\left(\frac{M-c}{M}\right)\left(\frac{M-c-\Delta s}{2}\right)(1-\underline{\pi}_A). \tag{29}$$

We now investigate how the peer quality difference Δs depends on $\underline{\pi}_A$. Next observe that the fraction of A households attending school A is

$$\left(\frac{M-c-\Delta s}{2M}\right)\underline{\pi}_A+\frac{M+c+\Delta s}{2M}$$

and thus the fraction of B households attending school A is

$$1-\left(\frac{M-c-\Delta s}{2M}\right)\underline{\pi}_A-\frac{M+c+\Delta s}{2M}.$$

Thus, the peer difference is

$$\Delta s=2\mu\left[\frac{M+c+\Delta s}{M}+\left(\frac{M-c-\Delta s}{M}\right)\underline{\pi}_A-1\right].$$

This implies that

$$\Delta s\left(1-\frac{2\mu}{M}(1-\underline{\pi}_A)\right)=2\mu\left[\frac{M+c}{M}+\left(\frac{M-c}{M}\right)\underline{\pi}_A-1\right],$$

which implies that

$$\Delta s\left(M-2\mu\left(1-\underline{\pi}_A\right)\right)=2\mu\left[c+\left(M-c\right)\underline{\pi}_A\right].$$

Since $\Delta s\geq 0$ and the right hand side is positive, this equation implies that $M>2\mu\left(1-\underline{\pi}_A\right)$ and therefore that

$$\Delta s=\frac{2\mu\left[c+\left(M-c\right)\underline{\pi}_A\right]}{M-2\mu\left(1-\underline{\pi}_A\right)}.$$

Given this, the fact that $M>c+\Delta s$, implies that

$$M-c>\frac{2\mu\left[c+\left(M-c\right)\underline{\pi}_A\right]}{M-2\mu\left(1-\underline{\pi}_A\right)}.$$

This requires that $M > c + 2\mu$.

Now observe that

$$\begin{aligned}
\frac{d\Delta s}{d\underline{\pi}_A} &= \frac{2\mu (M-c)(M-2\mu(1-\underline{\pi}_A)) - (2\mu)^2 [c + (M-c)\underline{\pi}_A]}{(M-2\mu(1-\underline{\pi}_A))^2} \\
&= \frac{2\mu (M-c)(M-2\mu) + (2\mu)^2 (M-c)\underline{\pi}_A - (2\mu)^2 [c + (M-c)\underline{\pi}_A]}{(M-2\mu(1-\underline{\pi}_A))^2} \\
&= \frac{2\mu (M-c)(M-2\mu) - (2\mu)^2 c}{(M-2\mu(1-\underline{\pi}_A))^2} \\
&= \frac{2\mu M (M-2\mu) - c2\mu (M-2\mu) - (2\mu)^2 c}{(M-2\mu(1-\underline{\pi}_A))^2} \\
&= \frac{2\mu M (M-2\mu-c)}{(M-2\mu(1-\underline{\pi}_A))^2} > 0
\end{aligned}$$

Thus, the peer quality difference is an increasing function of $\underline{\pi}_A$.

Now suppose the school district changed the assignment by decreasing $\underline{\pi}_A$ marginally and adjusting $\underline{\pi}_B$ to maintain feasibility. Given that the peer difference would decrease, it is clear from (29) that this would raise the school district's payoff. It follows that $\{(1, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ could not be a second best assignment.

B.7: Proof of Lemma 5

Suppose that $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ is a second best assignment such that $M > -c + \Delta s$ and $\underline{\pi}_B > 0$. By Lemma 3, we know that $\Delta s \geq 0$. It follows that $M > c - \Delta s$. By Lemma 2, it follows that $\overline{\pi}_B = 1$. In addition, we know by Lemma 4 that either $M < c + \Delta s$ or that $\underline{\pi}_A = 0$. Since when $M < c + \Delta s$, the value of $\underline{\pi}_A$ is irrelevant, we can without loss of generality set it to 0. Consider changing the assignment by reducing $\underline{\pi}_B$ to zero and raising $\overline{\pi}_A$ to keep school A full. Let $\overline{\pi}'_A > \overline{\pi}_A$ be the new probability that A households are admitted and let $\Delta s' > \Delta s$ be the new peer quality difference.

There are two cases to consider: i) $M - c > \Delta s$ and ii) $M - c \leq \Delta s$. In case i), some A households want to attend school B in the original assignment, and, in case ii), all A households want to attend school A. We begin with case i).

Panel 1 of Figure C7 illustrates the assignment of household types to schools under the original policy and Panel 2 illustrates the assignment under the new policy. The latter assumes that $M - c > \Delta s'$, so that some A households continue to want to attend school B in the new assignment.

What happens when we move from the original to the new policy? All the B households for whom $m \leq c - \Delta s'$ no longer attend school A. As illustrated by the Figure, they are replaced by three groups of households.

1. The first group are B households with $m \in [c - \Delta s', c - \Delta s]$. The probability that these households attend school A is raised from $\underline{\pi}_B$ to 1.

2. The second group are A households with $m \in [-c-\Delta s, M]$. The probability that these households attend school A is raised from $\overline{\pi}_A$ to $\overline{\pi}'_A$.

3. The third group are A households with $m \in [-c-\Delta s', -c-\Delta s]$. The probability that these households attend school A is raised from 0 to $\overline{\pi}'_A$.

Intuitively, it seems like there is a trade-off involved with the new policy. It is better to have the B households for whom $m \leq c - \Delta s'$ all attend school B. It is also better to have the second group attend school A with higher probability. However, groups 1 and 3 are further misassigned under the new policy. Nonetheless, it can be shown that the expected benefits of optimally assigning the B households for whom $m \leq c - \Delta s'$ exceed the expected costs associated with distorting groups 1 and 3.

To be more precise, randomly draw a B household for whom $m \leq c-\Delta s'$ who was assigned to school A under the original policy. The welfare gain from reassigning this household to school B is $c - m$. The expected gain is therefore

$$c - E(m|\, m \leq c - \Delta s') = c - \frac{c - \Delta s' - M}{2} = \frac{M + c + \Delta s'}{2}.$$

This household's seat in school A is taken by a household from one of the three groups identified above.

Suppose that the replacement household is from group 1. Then, if this household has type m', the welfare gain (which will be negative) from moving the household to school A is $m' - c$. The expected gain is therefore

$$E(m'|\, m' \in [c - \Delta s', c - \Delta s]) - c = c - \frac{\Delta s}{2} - \frac{\Delta s'}{2} - c = -\frac{\Delta s}{2} - \frac{\Delta s'}{2}.$$

The total expected gain from the reassignment of households is therefore

$$\frac{M + c - \Delta s}{2}.$$

This is positive since $M - c > \Delta s$ by hypothesis.

Suppose that the replacement household is from group 2. Then, if this household has type m', the welfare gain (which will be negative) from moving the household to school A is $m' - (-c)$. The expected gain is therefore

$$E(m'|\, m' \in [-c - \Delta s, M]) + c = \frac{M - c - \Delta s}{2} + c = \frac{M + c - \Delta s}{2}.$$

The total expected gain from the reallocation of households is therefore

$$M + c + \frac{\Delta s' - \Delta s}{2}.$$

This is positive.

Suppose that the replacement household is from group 3. Then, if this household has type m', the welfare gain (which will be negative) from moving the household to school A is $m' - (-c)$. The expected gain is therefore

$$E(m' | \, m' \in [-c - \Delta s', -c - \Delta s]) + c = -c - \frac{\Delta s}{2} - \frac{\Delta s'}{2} + c = -\frac{\Delta s}{2} - \frac{\Delta s'}{2}.$$

The total expected gain from the reassignment of households is therefore

$$\frac{M + c - \Delta s}{2}.$$

This is positive, as argued above.

It follows from this that the new policy increases welfare, which means that the original assignment could not have been second best.

As noted, this analysis assumes that $M - c > \Delta s'$, so that some A households continue to want to attend school B in the new assignment. What happens if $M - c \leq \Delta s'$ so that all A households continue to want to attend school B in the new assignment?

The only difference in the analysis is that the third group are now A households with $m \in [-M, -c - \Delta s]$. The expected gain from reassigning one of these households from school B to school A is therefore

$$E(m' | \, m' \in [-M, -c - \Delta s]) + c = -\frac{(M + c + \Delta s)}{2} + c = -\frac{(M - c + \Delta s)}{2}.$$

The total expected gain from the reassignment of households is therefore

$$\frac{M + c + \Delta s'}{2} - \frac{(M - c + \Delta s)}{2} = c + \frac{\Delta s' - \Delta s}{2}.$$

This is positive.

Now consider case ii). Recall that this is when $M - c \leq \Delta s$, so that all A households want to attend school A in the original assignment. Since $\Delta s' > \Delta s$, all A households want to attend school A in the new assignment as well. The situation is illustrated in Figure C8. Panel 1 illustrates the assignment of household types to schools under the original policy and Panel 2 illustrates the assignment under the new policy.

The differences are twofold: first, the second group consists of the entire group of A households, so they have $m \in [-M, M]$, and, second, the third group disappears. Accordingly, if the replacement household is from group 2, the expected gain is

$$E(m' | \, m' \in [-M, M]) + c = 0 + c = c.$$

The total expected gain from the reassignment of households is therefore

$$\frac{M + c + \Delta s'}{2} + c > 0.$$

If the replacement household is from group 1, the total expected gain from the reassignment of households continues to be

$$\frac{M + c - \Delta s}{2}.$$

However, in this case, we can no longer use the fact that $M - c > \Delta s$ to conclude that this is positive. Nonetheless, we can show directly that $M + c > \Delta s$.

Since $\Delta s' > \Delta s$, it is enough to show that $M + c > \Delta s'$. The fraction of B households assigned to school A in the new assignment is

$$\frac{M - c + \Delta s'}{2M}.$$

Feasibility implies that the fraction of A households assigned to school A in the new assignment is

$$1 - \frac{M - c + \Delta s'}{2M}.$$

Thus,

$$\Delta s' = 2\mu \left[1 - 2\left(\frac{M - c + \Delta s'}{2M}\right)\right].$$

This implies that

$$\Delta s' = 2\mu \left[\frac{c - \Delta s'}{M}\right].$$

It follows that

$$\Delta s'(1 + \frac{2\mu}{M}) = 2\mu \left[\frac{c}{M}\right] \Rightarrow \Delta s' = \frac{2\mu c}{M + 2\mu}.$$

This means that

$$M + c > \Delta s' \Leftrightarrow (M + c)(M + 2\mu) > 2\mu c,$$

which is true.

We conclude that the new policy also increases welfare in case ii), which means that the original assignment could not have been second best.

B.8: Proof of Lemma 6

Let $\{(\overline{\pi}_A, \underline{\pi}_A, \overline{\pi}_B, \underline{\pi}_B), \Delta s\}$ be a second best assignment. If the claim is not true, then both $\overline{\pi}_A$ and $\overline{\pi}_B$ must be less than 1. By Lemma 3, we can assume that $\Delta s \geq 0$. We can also assume with no loss of generality that $\underline{\pi}_A = 0$ and $\underline{\pi}_B = 0$. Lemma 2 tells us that $\underline{\pi}_A = 0$ if $M > c + \Delta s$. But if $M \leq c + \Delta s$ all A households want to attend school A and $\underline{\pi}_A$ is irrelevant, as long as it is less than $\overline{\pi}_A$. Accordingly, we can set it equal to 0. Similarly, Lemma 2 tells us that $\underline{\pi}_B = 0$ if $M > -c + \Delta s$. But if $M \leq -c + \Delta s$ all B households want to attend school A and $\underline{\pi}_B$ is irrelevant as long as it is less than $\overline{\pi}_B$.

We can also assume that at least some neighborhood B households would like to attend school A. If not, then $M + \Delta s \leq c$ which implies that $-M + \Delta s \geq -c$. This in turn implies that all neighborhood A households would like to attend school A. Since we know that $\underline{\pi}_B = 0$, this implies that $\overline{\pi}_A = 1$ to fill school A.

There are two possibilities to consider. The first is that all neighborhood A households would like to attend school A ($M \leq c + \Delta s$). In this case, the school district's payoff is

$$(1 - \overline{\pi}_A)(-c) + \frac{M + \Delta s - c}{2M}\overline{\pi}_B \left(\frac{M + c - \Delta s}{2} - c\right).$$

This equals

$$-(1-\overline{\pi}_A)c + \frac{\overline{\pi}_B}{4M}\left((M-c)^2 - \Delta s^2\right),$$

which is negative since $\Delta s \geq M - c$. This assignment therefore yields a lower payoff than neighborhood assignment and therefore cannot be second best.

The second possibility is that some neighborhood A households would like to attend school B ($M > c + \Delta s$). The school district's payoff in this case is

$$\frac{M+\Delta s+c}{2M}\left(\overline{\pi}_A\left(\frac{M-c-\Delta s}{2}\right) + (1-\overline{\pi}_A)(-c)\right)$$
$$+\frac{M-\Delta s-c}{2M}(-c) + \frac{M+\Delta s-c}{2M}\overline{\pi}_B\left(\frac{M+c-\Delta s}{2} - c\right).$$

This can be written more compactly as

$$\frac{M+\Delta s+c}{2M}\overline{\pi}_A\left(\frac{M+c-\Delta s}{2}\right) - c + \frac{M+\Delta s-c}{2M}\overline{\pi}_B\left(\frac{M-c-\Delta s}{2}\right). \tag{30}$$

We will now write this payoff as a function of $\overline{\pi}_B$ and show that marginally changing $\overline{\pi}_B$ can create an increase in the school district's payoff. The first task is to develop an expression for Δs as a function of $\overline{\pi}_B$. The fraction of B households in school A is

$$\left(\frac{M+\Delta s-c}{2M}\right)\overline{\pi}_B,$$

and the fraction of A households in school A is

$$\left(\frac{M+\Delta s+c}{2M}\right)\overline{\pi}_A.$$

The feasibility constraint implies that

$$\left(\frac{M+\Delta s+c}{2M}\right)\overline{\pi}_A = 1 - \left(\frac{M+\Delta s-c}{2M}\right)\overline{\pi}_B. \tag{31}$$

Substituting this into (30) and rearranging, we can write the payoff as

$$\frac{M-\Delta s}{2} - c\left(\frac{1}{2} + \left(\frac{M+\Delta s-c}{2M}\right)\overline{\pi}_B\right). \tag{32}$$

Furthermore, we can write the peer quality difference as:

$$\Delta s = 2\mu\left(1 - 2\overline{\pi}_B\left(\frac{M+\Delta s-c}{2M}\right)\right) = 2\mu\left(\frac{M-\overline{\pi}_B(M+\Delta s-c)}{M}\right).$$

Solving for Δs yields

$$\Delta s = 2\mu\left(\frac{M-(M-c)\overline{\pi}_B}{M+2\mu\overline{\pi}_B}\right). \tag{33}$$

From (33), the derivative of the peer quality difference is

$$\begin{aligned}
\frac{d\Delta s}{d\overline{\pi}_B} &= -2\mu\left(\frac{(M+2\mu\overline{\pi}_B)(M-c)-2\mu(M-c)\overline{\pi}_B+2\mu M}{(M+2\mu\overline{\pi}_B)^2}\right) \\
&= -2\mu M\left(\frac{M-c+2\mu}{(M+2\mu\overline{\pi}_B)^2}\right).
\end{aligned}$$

This is negative, which makes sense intuitively.

Now consider the welfare consequences of a marginal change in $\overline{\pi}_B$. From (32), the derivative of the school district's payoff is:

$$\begin{aligned}
& -\frac{d\Delta s/d\overline{\pi}_B}{2}-c\left(\left(\frac{M+\Delta s-c}{2M}\right)+\left(\frac{d\Delta s/d\overline{\pi}_B}{2M}\right)\overline{\pi}_B\right) \\
= & -\frac{1}{2M}\left[c\,(M+\Delta s-c)+\frac{d\Delta s}{d\overline{\pi}_B}(M+c\overline{\pi}_B)\right].
\end{aligned}$$

Substituting in for Δs and $d\Delta s/d\overline{\pi}_B$, we have

$$\begin{aligned}
& c\,(M+\Delta s-c)+\frac{d\Delta s}{d\overline{\pi}_B}(M+c\overline{\pi}_B) \\
= & c\left(M+2\mu\left(\frac{M-(M-c)\overline{\pi}_B}{M+2\mu\overline{\pi}_B}\right)-c\right)-2\mu M\left(\frac{M-c+2\mu}{(M+2\mu\overline{\pi}_B)^2}\right)(M+c\overline{\pi}_B) \\
= & c\left(\frac{-2\mu(M-c)\overline{\pi}_B+2\mu M+(M-c)(M+2\mu\overline{\pi}_B)}{M+2\mu\overline{\pi}_B}\right)-2\mu M\left(\frac{M-c+2\mu}{(M+2\mu\overline{\pi}_B)^2}\right)(M+c\overline{\pi}_B) \\
= & cM\left(\frac{M-c+2\mu}{M+2\mu\overline{\pi}_B}\right)-2\mu M\left(\frac{M-c+2\mu}{(M+2\mu\overline{\pi}_B)^2}\right)(M+c\overline{\pi}_B) \\
= & M\left(\frac{M-c+2\mu}{M+2\mu\overline{\pi}_B}\right)\left[c-\frac{2\mu\,(M+c\overline{\pi}_B)}{M+2\mu\overline{\pi}_B}\right] \\
= & M\left(\frac{M-c+2\mu}{M+2\mu\overline{\pi}_B}\right)\left[\frac{(M+2\mu\overline{\pi}_B)\,c-2\mu\,(M+c\overline{\pi}_B)}{M+2\mu\overline{\pi}_B}\right] \\
= & M^2\left(\frac{M-c+2\mu}{(M+2\mu\overline{\pi}_B)^2}\right)[c-2\mu]
\end{aligned}$$

Thus, the derivative of the school district's payoff is

$$-\frac{M}{2}\left(\frac{M-c+2\mu}{(M+2\mu\overline{\pi}_B)^2}\right)(c-2\mu).$$

This derivative is either positive or negative if $c\neq 2\mu$. Thus, either a small increase or decrease in $\overline{\pi}_B$ will increase welfare. The assignment cannot therefore be second best.

B.9: Proof of Lemma 7

There are two possibilities to consider. The first is that $M\leq c+\Delta s$. In this case, all A households wish to attend school A. Given that $\overline{\pi}_A=1$, all A households are assigned to school A. Feasibility therefore requires that $\overline{\pi}_B$ and hence $\underline{\pi}_B$ must equal 0. It does not matter what $\underline{\pi}_A$ is because no A households wish to apply to school B. Given that school A consists only of A students, we have that $\Delta s = 2\mu$. Accordingly, for this possibility to arise, we require that $M - c \leq 2\mu$.

The second possibility is that $M > \Delta s + c$. In this case, some A households wish to attend school B and some B households wish to attend school A. Given that $\overline{\pi}_A = 1$, the fraction of A households assigned to school A is

$$\frac{M + \Delta s + c}{2M}.$$

The fraction of B households who would like to attend school A is

$$\frac{M + \Delta s - c}{2M}.$$

Given that $\Delta s > 0$, feasibility therefore requires that $\overline{\pi}_B < 1$. Lemma 4 implies that $\underline{\pi}_A = 0$ and Lemma 2 implies that $\underline{\pi}_B = 0$ (since $\overline{\pi}_B < 1$). Thus, from the feasibility constraint, we have that $\overline{\pi}_B$ satisfies.

$$\frac{M + \Delta s + c}{2M} + \overline{\pi}_B \left(\frac{M + \Delta s - c}{2M} \right) = 1.$$

This implies that

$$M + \Delta s + c + \overline{\pi}_B \left(M + \Delta s - c \right) = 2M,$$

which means that

$$\overline{\pi}_B = \frac{M - \Delta s - c}{M + \Delta s - c}.$$

In addition, we have that

$$\begin{aligned}
\Delta s &= 2\mu \left(\frac{M + \Delta s + c}{2M} - \overline{\pi}_B \left(\frac{M + \Delta s - c}{2M} \right) \right) \\
&= 2\mu \left(\frac{M + \Delta s + c}{2M} - \frac{M - \Delta s - c}{2M} \right) \\
&= 2\mu \left(\frac{\Delta s + c}{M} \right).
\end{aligned}$$

Thus

$$\Delta s = \frac{2\mu c}{M - 2\mu},$$

and

$$\overline{\pi}_B = \frac{M - c - \frac{2\mu c}{M - 2\mu}}{M - c + \frac{2\mu c}{M - 2\mu}}.$$

In order for this possibility, we therefore require that

$$M > \frac{2\mu c}{M - 2\mu} + c,$$

which is equivalent to

$$M > c + 2\mu.$$

B.10: Proof of Lemma 8

If $M \leq c+2\mu$, then by Lemma 7, all A households wish to attend school A and since $\overline{\pi}_A = 1$, they are all assigned to school A. All B households are assigned to school B so that the assignment corresponds to neighborhood assignment. This generates a payoff of 0 for the school district.

If $M > c + 2\mu$, then by Lemma 7, some A households wish to attend school B and since $\underline{\pi}_A = 0$, they are all assigned to school B. Some B households wish to attend school A and are assigned to it with probability $\overline{\pi}_B$. The school district's payoff is

$$\frac{M+\Delta s+c}{2M}\left(\frac{M-c-\Delta s}{2}\right)+\frac{M-\Delta s-c}{2M}(-c)$$
$$+\overline{\pi}_B\left(\frac{M-c+\Delta s}{2M}\right)\left(\frac{M+c-\Delta s}{2}-c\right).$$

This simplifies to

$$(1+\overline{\pi}_B)\left(\frac{M-c+\Delta s}{2M}\right)\frac{M-c-\Delta s}{2}.$$

Lemma 7 tells us that

$$\overline{\pi}_B = \frac{M-c-\Delta s}{M-c+\Delta s},$$

so this equals

$$\frac{2\,(M-c)}{M-c+\Delta s}\left(\frac{M-c+\Delta s}{2M}\right)\frac{M-c-\Delta s}{2}.$$

This reduces to the expression presented in the statement of Lemma 8.

B.11: Proof of Lemma 9

There are three possibilities to consider. The first is that $M + c \leq \Delta s$. In this case, all households wish to attend school A. Given that B households have priority, this means that school A consists entirely of B students. School A therefore consists entirely of B students, so that $\Delta s = -2\mu$. It is therefore not possible that $M + c \leq \Delta s$.

The next possibility is that $M+c > \Delta s \geq M-c$. In this case, all A households and some B households wish to attend school A. Given that $\overline{\pi}_B = 1$, Lemma 5 implies that $\underline{\pi}_B = 0$. This means that the fraction of B households in school A is

$$\frac{M+\Delta s-c}{2M}.$$

It follows that

$$\overline{\pi}_A = 1-\frac{M+\Delta s-c}{2M} = \frac{M-\Delta s+c}{2M}.$$

It does not matter what $\underline{\pi}_A$ is, as long as it is less than $\overline{\pi}_A$, because no neighborhood A households wish to apply to school B.

It follows from all this that

$$\begin{aligned}\Delta s &= 2\mu\left(\frac{M-\Delta s+c}{2M}-\frac{M+\Delta s-c}{2M}\right)\\ &= 2\mu\left(\frac{c-\Delta s}{M}\right).\end{aligned}$$

Thus

$$\Delta s=\frac{2\mu c}{M+2\mu}$$

and

$$\overline{\pi}_A=\frac{M+c-\frac{2\mu c}{M+2\mu}}{2M}.$$

In order for $M-c\leq\Delta s$, we therefore need that

$$M\leq c+\frac{2\mu c}{M+2\mu},$$

which is equivalent to

$$M\left(M+2\mu-c\right)\leq 4c\mu.$$

The equation

$$M\left(M+2\mu-c\right)=4c\mu,$$

is a quadratic equation which has solutions

$$\begin{aligned}M &= \frac{c-2\mu\pm\sqrt{(2\mu-c)^2+16c\mu}}{2}\\ &= \frac{c-2\mu\pm\sqrt{(2\mu+c)^2+8c\mu}}{2}.\end{aligned}$$

The positive root is

$$M=\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2}.$$

The condition that $M\left(M+2\mu-c\right)\leq 4c\mu$, therefore amounts to the requirement that

$$M\leq\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2}.$$

The third possibility is that $M>\Delta s+c$. In this case, some A households wish to attend school B and some B households wish to attend school A. Given that $\overline{\pi}_B=1$, Lemma 5 implies that $\underline{\pi}_B=0$. This means that the fraction of B households in school A is

$$\frac{M+\Delta s-c}{2M}.$$

The fraction of A households wishing to attend school A is

$$\frac{M+\Delta s+c}{2M}.$$

Since $\Delta s>0$, feasibility therefore requires that $\overline{\pi}_A<1$. Lemma 2 then implies that $\underline{\pi}_A=0$. This means that $\overline{\pi}_A$ satisfies.

$$\overline{\pi}_A\left(\frac{M+\Delta s+c}{2M}\right)+\frac{M+\Delta s-c}{2M}=1.$$

This implies that

$$M+\Delta s-c+\overline{\pi}_A\left(M+\Delta s+c\right)=2M,$$

which means that

$$\overline{\pi}_A=\frac{M-\Delta s+c}{M+\Delta s+c}.$$

In addition, we have that

$$\begin{aligned}
\Delta s &= 2\mu\left(\overline{\pi}_A\left(\frac{M+\Delta s+c}{2M}\right)-\frac{M+\Delta s-c}{2M}\right)\\
&= 2\mu\left(\frac{M-\Delta s+c}{2M}-\frac{M+\Delta s-c}{2M}\right)\\
&= 2\mu\left(\frac{c-\Delta s}{M}\right).
\end{aligned}$$

Thus

$$\Delta s=\frac{2\mu c}{M+2\mu}.$$

and

$$\overline{\pi}_A=\frac{M+c-\frac{2\mu c}{M+2\mu}}{M+c+\frac{2\mu c}{M+2\mu}}$$

In order that $M>\Delta s+c$, we require that

$$M>\frac{2\mu c}{M+2\mu}+c,$$

which is equivalent to

$$M(M+2\mu-c)>4c\mu.$$

As shown above, this requires that

$$M>\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2}.$$

B.12: Proof of Lemma 10

In the case in which $\Delta s \geq M - c$, the school district's payoff is

$$\overline{\pi}_A(0) + (1-\overline{\pi}_A)(-c) + \frac{M+c-\Delta s}{2M}(0) + \frac{M-c+\Delta s}{2M}\left(\frac{M+c-\Delta s}{2} - c\right)$$

We have that

$$1 - \overline{\pi}_A = \frac{M-c+\Delta s}{2M},$$

so this simplifies to

$$\frac{M-c+\Delta s}{2M}\left(\frac{M-c-\Delta s}{2} - c\right).$$

As shown in Lemma 9, this case arises when

$$M \leq \frac{c - 2\mu + \sqrt{(2\mu+c)^2 + 8c\mu}}{2},$$

and in this case

$$\Delta s = \frac{2\mu c}{M + 2\mu}.$$

In the case in which $\Delta s < M - c$, the school district's payoff is

$$\frac{M-\Delta s-c}{2M}(-c) + \frac{M+\Delta s+c}{2M}\left(\overline{\pi}_A(\frac{M-c-\Delta s}{2}) + (1-\overline{\pi}_A)(-c)\right) +$$
$$\frac{M+c-\Delta s}{2M}(0) + \frac{M-c+\Delta s}{2M}\left(\frac{M+c-\Delta s}{2} - c\right).$$

This equals

$$-c + \frac{M+\Delta s+c}{2M}\overline{\pi}_A(\frac{M+c-\Delta s}{2}) + \frac{M-c+\Delta s}{2M}\left(\frac{M-c-\Delta s}{2}\right).$$

We have that

$$\overline{\pi}_A\left(\frac{M+\Delta s+c}{2M}\right) = 1 - \frac{M+\Delta s-c}{2M} = \frac{M-\Delta s+c}{2M},$$

so the school district's payoff is

$$-c + \frac{M-\Delta s+c}{2M}(\frac{M+c-\Delta s}{2}) + \frac{M-c+\Delta s}{2M}\left(\frac{M-c-\Delta s}{2}\right).$$

This reduces to

$$-c + \frac{M^2 + c^2 - \Delta s(M+c)}{2M}.$$

As shown in Lemma 10, this case arises when

$$M > \frac{c - 2\mu + \sqrt{(2\mu+c)^2 + 8c\mu}}{2},$$

and in this case

$$\Delta s = \frac{2\mu c}{M + 2\mu}.$$

B13: Proof of Lemma 11

We begin with part i) and thus assume that $c > 2\mu$. Suppose first that $M < \frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2}$. It is straightforward to verify that

$$\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2} < c+2\mu.$$

Thus, from (19) and (22), we have that

$$W_A = 0,$$

and that

$$W_B = \left(\frac{M+\Delta s_B - c}{2M}\right)\left(\frac{M-\Delta s_B - c}{2M} - c\right).$$

But, since M is less than $\left(c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}\right)/2$, we know from the proof of Lemma 9 that $M - \Delta s_B - c < 0$. This implies that $W_B < 0$, which means that A priority yields a higher payoff.

Next suppose that $M \geq \left(c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}\right)/2$. There are two cases to consider. The first is when $M > c+2\mu$. In this case, it follows from (19) and (22) that A priority yields a higher payoff than B priority, when

$$\frac{M^2+c^2-(M+c)\,\Delta s_B}{2M} < \frac{M^2+c^2-(M-c)\,\Delta s_A}{2M}.$$

This is equivalent to

$$\begin{array}{rcl}
(M-c)\,\Delta s_A & < & (M+c)\,\Delta s_B \\
\dfrac{(M-c)\,2\mu c}{M-2\mu} & < & \dfrac{(M+c)\,2\mu c}{M+2\mu} \\
(M-c)\,(M+2\mu) & < & (M+c)\,(M-2\mu) \\
M^2+2\mu M - cM - c\mu 2 & < & M^2 - 2\mu M + cM - c\mu 2 \\
2\mu & < & c.
\end{array}$$

Thus, since $2\mu < c$, A priority yields a higher payoff than B priority.

The second case is when $\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2} \leq M \leq c+2\mu$. In this case, it follows from (19) and (22) that A priority yields a higher payoff than B priority, when

$$\begin{array}{rcl}
\dfrac{M^2+c^2-(M+c)\,\Delta s_B}{2M} & < & c \\
M^2+c^2-2Mc & < & (M+c)\,\Delta s_B \\
M^2+c^2-2Mc & < & \dfrac{(M+c)\,2\mu c}{M+2\mu} \\
M^3+c^2M-2M^2c+M^2 2\mu + c^2 2\mu - 4Mc\mu & < & M2\mu c + 2\mu c^2 \\
M^3+c^2M+M^2 2\mu & < & 6M\mu c + 2M^2 c \\
M^2+c^2+M2\mu & < & 6\mu c + 2Mc \\
M^2+M2\,(\mu-c)+c\,(c-6\mu) & < & 0
\end{array}$$

To convert this into a a cleaner condition on M, we need to solve the quadratic equation

$$M^2 + M2(\mu - c) - c(6\mu - c) = 0.$$

This has solution

$$M = \frac{2c - 2\mu \pm \sqrt{(2\mu - 2c)^2 + 4c(6\mu - c)}}{2}.$$

The relevant root is the positive one

$$\begin{aligned} & \frac{2c - 2\mu + \sqrt{(2\mu - 2c)^2 + 4c(6\mu - c)}}{2} \\ = \; & \frac{2c - 2\mu + \sqrt{4\mu^2 + 4c^2 - 8\mu c + 24c\mu - 4c^2}}{2} \\ = \; & \frac{2c - 2\mu + \sqrt{4\mu^2 + 16c\mu}}{2} \\ = \; & c - \mu + \sqrt{\mu^2 + 4c\mu}. \end{aligned}$$

Thus, we conclude that, in the case $\frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2} \leq M \leq c + 2\mu$, A priority dominates B priority when

$$M < c - \mu + \sqrt{\mu^2 + 4c\mu}.$$

Observe that

$$\begin{aligned} c - \mu + \sqrt{\mu^2 + 4c\mu} \quad & > \quad c + 2\mu \\ & \iff \quad \sqrt{\mu^2 + 4c\mu} > 3\mu \\ & \iff \quad \mu^2 + 4c\mu > 9\mu^2 \\ & \iff \quad c > 2\mu \end{aligned}$$

Thus, when $c > 2\mu$, there is no range in which B priority dominates A priority.

We now turn to part ii) of the Lemma and thus assume that $c < 2\mu$. Then, as just argued, we have that

$$c - \mu + \sqrt{\mu^2 + 4c\mu} < c + 2\mu.$$

Moreover, we have that

$$c - \mu + \sqrt{\mu^2 + 4c\mu} > \frac{c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}}{2}$$

To see this note that this inequality is equivalent to

$$\begin{aligned} 2c - 2\mu + 2\sqrt{\mu^2 + 4c\mu} \quad & > \quad c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu} \\ c + \sqrt{4\mu^2 + 16c\mu} \quad & > \quad \sqrt{(2\mu + c)^2 + 8c\mu} \end{aligned}$$

The latter is true if $c < 2\mu$.

Thus, suppose that $M < c - \mu + \sqrt{\mu^2 + 4c\mu}$. Then, $M < c + 2\mu$ and it follows from (19) that W_A equals 0. From (22), we have that

$$W_B = \begin{cases} \left(\frac{M+\Delta s_B - c}{2M}\right)\left(\frac{M-\Delta s_B - c}{2M} - c\right) \text{ if } M < \frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2} \\ \frac{(M-c)^2-(M+c)\Delta s_B}{2M} \text{ if } M > \frac{c-2\mu+\sqrt{(2\mu+c)^2+8c\mu}}{2} \end{cases}.$$

We already know that $W_B < 0$ when $M < \left(c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}\right)/2$. Moreover, when the reverse inequality holds, we know that

$$\frac{(M-c)^2 - (M+c)\,\Delta s_B}{2M} = \frac{M^2 + c^2 - (M+c)\,\Delta s_B}{2M} - c < 0$$

when $M < c - \mu + \sqrt{\mu^2 + 4c\mu}$. Thus, A priority dominates B priority in this range.

Now suppose that $M > c - \mu + \sqrt{\mu^2 + 4c\mu}$. Then $M > \left(c - 2\mu + \sqrt{(2\mu + c)^2 + 8c\mu}\right)/2$, so that from (22), we have that

$$W_B = \frac{M^2 + c^2 - (M+c)\,\Delta s_B}{2M} - c.$$

We know that $W_B > 0$, so W_B exceeds W_A if $M < c + 2\mu$. If $M > c + 2\mu$, it follows from (19) that W_B exceeds W_A when

$$\frac{M^2 + c^2 - (M+c)\,\Delta s_B}{2M} > \frac{M^2 + c^2 - (M-c)\,\Delta s_A}{2M}.$$

This is equivalent to

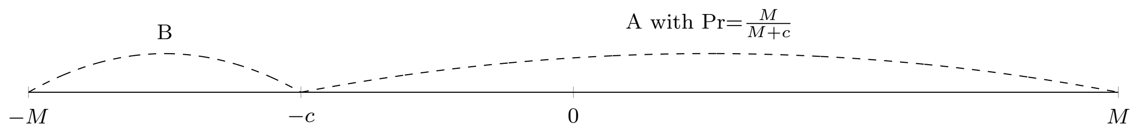
$$\begin{aligned} (M-c)\,\Delta s_A &> (M+c)\,\Delta s_B \\ \frac{(M-c)\,2\mu c}{M-2\mu} &> \frac{(M+c)\,2\mu c}{M+2\mu} \\ (M-c)\,(M+2\mu) &> (M+c)\,(M-2\mu) \\ M^2 + 2\mu M - cM - c\mu 2 &> M^2 - 2\mu M + cM - c\mu 2 \\ 4\mu M &> 2cM \\ 2\mu &> c. \end{aligned}$$

Thus, since $2\mu > c$, B priority yields a higher payoff than A priority. ■

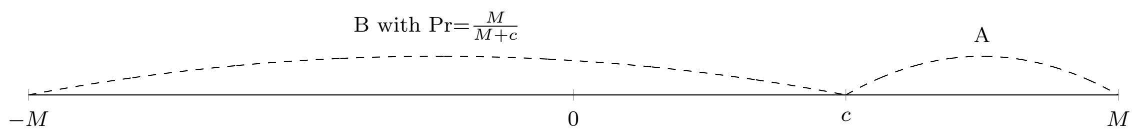
Appendix C. Figures for Appendix B

Figure C1

Assignment of neighborhood A households



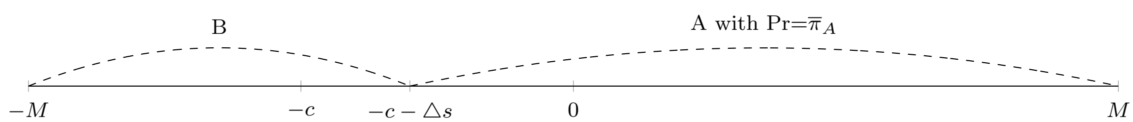
Assignment of neighborhood B households



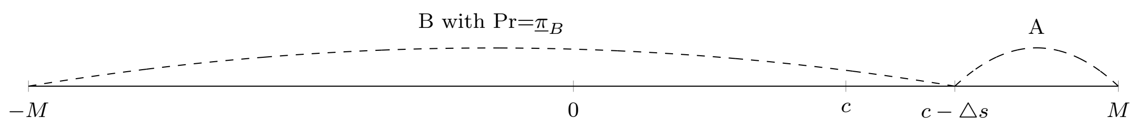
Note: Panel (a) describes how households in neighborhood A are allocated to schools as a function of their match value m. For example, households with $m \in (-c, M)$ are assigned to school A with probability $\frac{M}{M+c}$, while households with $m \in (-M, -c)$ are assigned to school B with certainty. Panel (b) provides the analogous information for households in neighborhood B.

Figure C2

Assignment of neighborhood A households



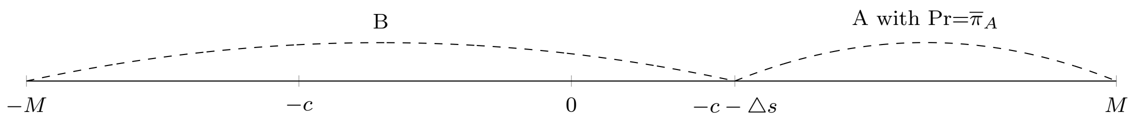
Assignment of neighborhood B households



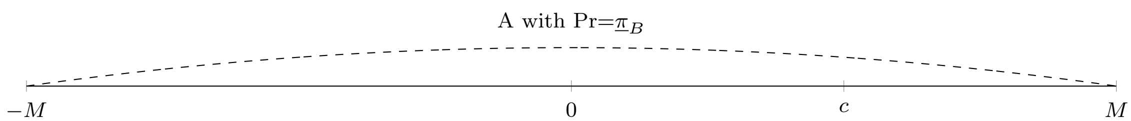
Note: See notes to Figure C1 and text for details.

Figure C3

Assignment of neighborhood A households



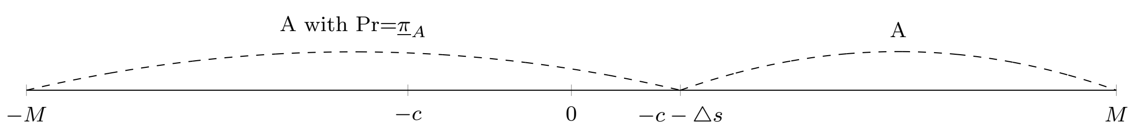
Assignment of neighborhood B households



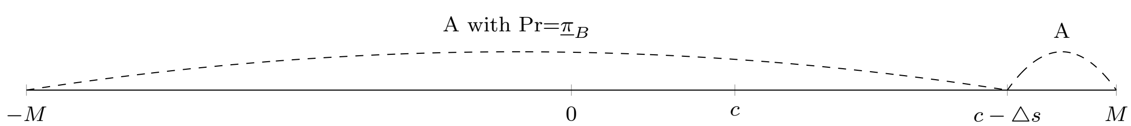
Note: See notes to Figure C1 and text for details.

Figure C4

Assignment of neighborhood A households



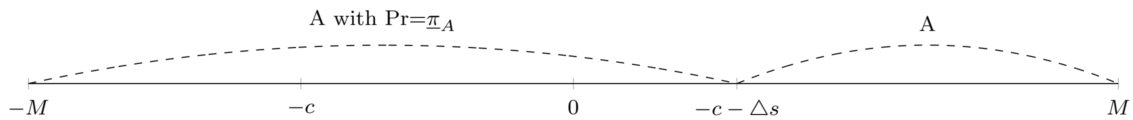
Assignment of neighborhood B households



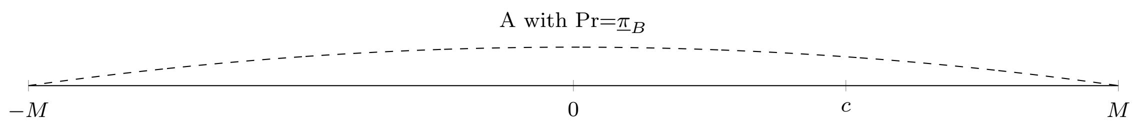
Note: See notes to Figure C1 and text for details.

Figure C5

Assignment of neighborhood A households



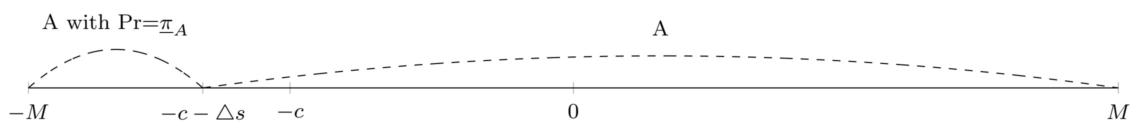
Assignment of neighborhood B households



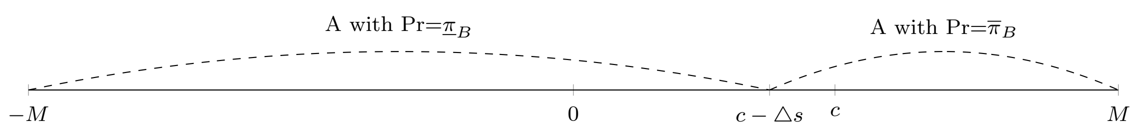
Note: See notes to Figure C1 and text for details.

Figure C6

Assignment of neighborhood A households



Assignment of neighborhood B households

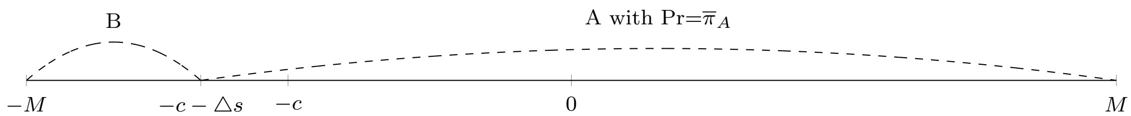


Note: See notes to Figure C1 and text for details.

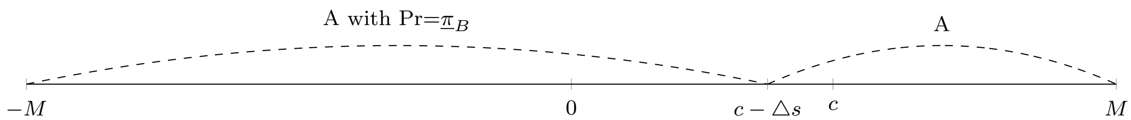
Figure C7

<u>Panel 1: Original policy</u>

Assignment of neighborhood A households

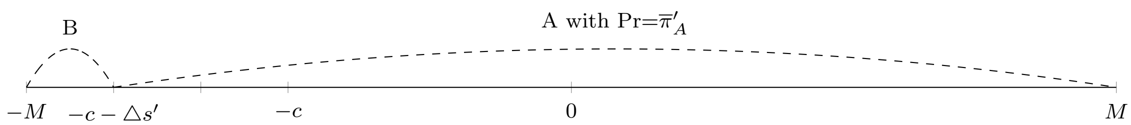


Assignment of neighborhood B households

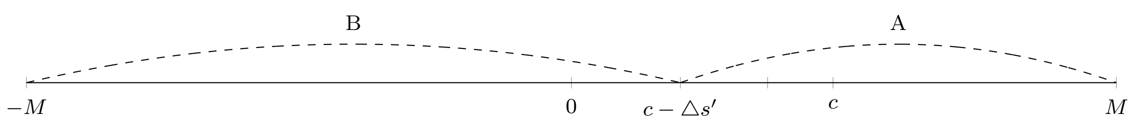


<u>Panel 2: New policy</u>

Assignment of neighborhood A households



Assignment of neighborhood B households

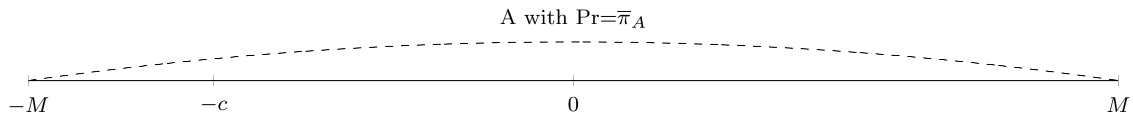


Note: See notes to Figure C1 and text for details.

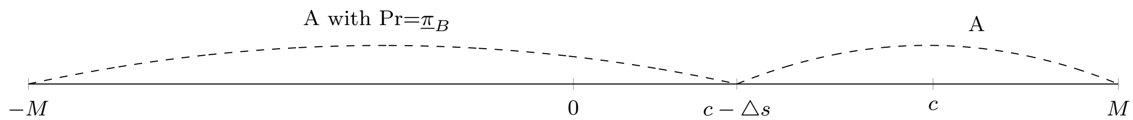
Figure C8

Panel 1: Original policy

Assignment of neighborhood A households

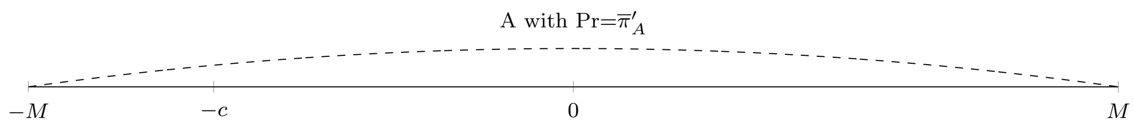


Assignment of neighborhood B households

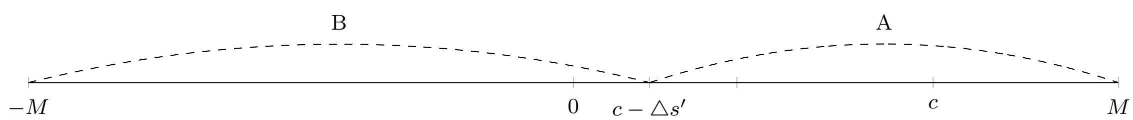


Panel 2: New policy

Assignment of neighborhood A households



Assignment of neighborhood B households



Note: See notes to Figure C1 and text for details.